

In-Order Sliding-Window Aggregation in Worst-Case Constant Time

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Abstract Sliding-window aggregation is a widely-used approach for extracting insights from the most recent portion of a data stream. The aggregations of interest can usually be expressed as binary operators that are associative but not necessarily commutative nor invertible. Non-invertible operators, however, are difficult to support efficiently. In a 2017 conference paper, we introduced DABA, the first algorithm for sliding-window aggregation with worst-case constant time. Before DABA, if a window had size n , the best published algorithms would require $O(\log n)$ aggregation steps per window operation—and while for strictly in-order streams, this bound could be improved to $O(1)$ aggregation steps on average, it was not known how to achieve an $O(1)$ bound for the worst-case, which is critical for latency-sensitive applications. This article is an extended version of our 2017 paper. Besides describing DABA in more detail, this article introduces a new variant, DABA Lite, which achieves the same time bounds in less memory. Whereas DABA requires space for storing $2n$ partial aggregates, DABA Lite only requires space for $n + 2$ partial aggregates. Our experiments on synthetic and real data support the theoretical findings.

Keywords Real-time, aggregation, continuous analytics, (de-)amortization

1 Introduction

Stream processing is a now-standard paradigm for handling high-speed continuous data, spurring the development of many stream-processing engines [2, 5, 6, 9, 13, 16, 23, 25, 37, 42]. Since stream processing is often subject to strict quality-of-service or real-time requirements, it requires low-latency

Table 1 Aggregation operators with their algebraic properties.

	Invertible	Associative	Commutative
Sum-like: sum, count, mean, geomean, stddev, ...	✓	✓	✓
Max-like: max, min, argMax, maxCount, M4 [20], ...	×	✓	×
Mergeable sketch [4]: Bloom, CountMin, HyperLogLog, algebraic classifiers [19], ...	×	✓	✓

responses. As a mainstay of stream processing, aggregation (e.g., computing the maximum, geometric mean, or more elaborate summaries such as Bloom filters [8]) is one of the most common computations in streaming applications, used both standalone and as a building block for other analytics. Unfortunately, existing techniques for sliding-window aggregation cannot consistently guarantee low latency.

Because the newest data is often deemed more pertinent or valuable than older data, streaming aggregation is typically performed on a sliding window (e.g., the last hour’s worth of data). This not only provides intuitive semantics to the end users but also helps bound the amount of data the system has to keep around. Following Boykin et al. [9], we use the term aggregation broadly, to include both classical relational aggregation operators such as sum, geometric mean, and maximum, as well as a more general class of associative operators. Table 1 lists several such operators and characterizes them by their algebraic properties. While some operators are invertible or commutative, many are not. This paper focuses on algorithms that work with all associative operators, including non-invertible and non-commutative ones.

An algorithm for sliding-window aggregation supports three operations (formally described in Section 2): *insert* for a data item’s arrival, *query* for requesting the current aggregation outcome, and *evict* for a data item’s depar-

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ture [18]. This paper presents the *De-Amortized Banker's Aggregator (DABA)*, a novel general-purpose sliding-window aggregation algorithm that guarantees low-latency response on every operation—in the worst case, not just on average. The algorithm is simple and supports both fixed-sized and variable-sized windows. It works as long as (i) the aggregation operator, denoted by \otimes in this paper, is an associative binary operator and (ii) the window has first-in first-out (FIFO) semantics. DABA does not require any other properties from the \otimes operator. In particular, DABA works equally well whether \otimes is invertible or non-invertible, commutative or non-commutative. DABA supports each of the *query*, *insert*, and *evict* operations by making at most a constant number of calls to the \otimes operator in the worst case. This is independent of the window size, denoted by n in this paper. If each invocation of \otimes takes constant time, then the DABA algorithm takes worst-case constant time.

We first published the DABA algorithm in a 2015 technical report [31] and later in a 2017 conference paper [32]. Prior to the publication of DABA, the algorithms with the best algorithmic complexity for this problem in the published literature took $O(\log n)$ time [7, 34], i.e., not $O(1)$ time like DABA. After the publication of DABA, there have been other papers with algorithms that take amortized $O(1)$ time for FIFO sliding-window aggregation [28, 33, 35, 40]. However, these algorithms maintain the $O(1)$ time complexity only in the amortized sense, i.e., not in the worst case like DABA. In terms of space complexity, for a window of size n , DABA stores $2n$ partial aggregates. This journal version of the DABA paper also introduces a new, previously unpublished algorithm called DABA Lite that reduces the memory requirements to $n + 2$ partial aggregates. Furthermore, this journal version has more extensive examples and visualizations for our algorithms.

The core idea behind DABA is to start from an algorithm that has amortized $O(1)$ time complexity and to de-amortize it. The algorithm that serves as a starting point is Two-Stacks. It uses an old trick from functional programming for representing a FIFO queue with two stacks, which we augment with aggregation. A special case of this algorithm was first mentioned in a Stack Overflow thread [3]. The Two-Stacks algorithm has one rare—but expensive—operation called *flip* that transfers all data from one stack to the other. The flip operation causes a latency spike, which can be undesirable for low-latency streaming. De-amortization turns the average-case $O(1)$ behavior of Two-Stacks into the worst-case $O(1)$ behavior of DABA by spreading out the expensive flip operation, thus eliminating the latency spike.

Two-Stacks, DABA, and DABA Lite only work for FIFO windows, i.e., for sliding windows over in-order streams. Handling out-of-order streams is beyond the scope of this paper. Indeed, it has been shown that the lower bound on the time complexity for aggregating out-of-order streams is

worse than $O(1)$ unless disorder in a stream is bounded by a constant [33]. Before the emergence of constant-time sliding window aggregation algorithms, a popular approach for achieving low latency was to use coarse-grained windows where evictions occur in batches. This approach reduces the effective window size n , thus making algorithms whose time complexity depends on n feasible. Pre-aggregating inside each batch reduces the cost of aggregating across batches [10, 22, 24]. But coarse-grained windows are an approximation that does not always satisfy application requirements. Another popular research topic in sliding-window aggregation is *window sharing*, where aggregations for multiple different window sizes are computed on a single data structure. While some sliding window aggregation algorithms support window sharing in amortized $O(1)$ time, none of them achieve worst-case $O(1)$ time [28, 33]. DABA and DABA Lite achieve worst-case $O(1)$ time but do not support window sharing.

Experiments show that DABA and DABA Lite perform well in practice. We have implemented our new algorithms in C++ and benchmarked them against average-case $O(1)$ algorithms. True to being worst-case $O(1)$, our results show that DABA and DABA Lite have lower latency and competitive throughput as we increase the window size. When the aggregation operation is cheap, the low latency and high throughput are due to constant-time updates to a lightweight data structure. When the aggregation operation is expensive, they are due to a low-constant number of calls to the costly aggregation operator.

Our implementations of DABA, DABA Lite, and all other algorithms used in this paper are available on GitHub from the open source project Sliding Window Aggregators¹.

2 Problem Definition

This section formalizes the problem of maintaining aggregation in a first-in first-out (FIFO) sliding window and discusses the kinds of aggregations supported in this work.

2.1 Sliding-Window Aggregation Data Type

This paper is concerned with sliding-window aggregation on in-order streams with a first-in first-out (FIFO) window. In this type of window, the earliest data item to arrive is also the earliest data item to leave the window. Hence, the sliding window is a queue that supports aggregation of its data. The front of the queue contains the earliest data, the back of the queue holds the latest data, and the aggregation is from the earliest to the latest. As a queue, the window is only affected by two kinds of changes:

¹ <https://github.com/IBM/sliding-window-aggregators>

Data Arrival: The arrival of a window data item results in a new data item at the end of the window. This is often triggered by the arrival of a data item in a relevant stream.

Data Eviction: An eviction causes the data item at the front of the window to be removed from the window. The choice of when this happens is typically controlled by the window policy (e.g., a time-based window evicts the earliest data item when it falls out of the time frame of interest and a count-based window evicts the earliest data item to keep the size fixed [15]). Window eviction policies are orthogonal to the algorithms in this paper.

We will model the problem of maintaining aggregation in a FIFO sliding window as an abstract data type (ADT) with an interface similar to that of a queue. To begin, we review an algebraic structure called a monoid:

Definition: A monoid is a triple $\mathcal{M} = (S, \otimes, \mathbf{1})$ with a binary operator $\otimes : S \times S \rightarrow S$ on S such that

- *Associativity:* For $a, b, c \in S$, $a \otimes (b \otimes c) = (a \otimes b) \otimes c$; and
- *Identity:* $\mathbf{1} \in S$ is the identity: $\mathbf{1} \otimes a = a = a \otimes \mathbf{1}$ for all $a \in S$.

In comparison to real-number arithmetic, the \otimes operator can be seen as a generalization of arithmetic multiplication where the identity element $\mathbf{1}$ is a generalization of the number 1. Some of our earlier papers instead used an analogy to arithmetic addition with an identity element of zero. While that works equally well, here we adopt the multiplication analogy, because it makes it natural to adopt a shorthand notation of abc for $a \otimes b \otimes c$. That shorthand makes it easier to write detailed examples.

A monoid is *commutative* if $a \otimes b = b \otimes a$ for all $a, b \in S$. A monoid has a *left inverse* if there exists a (known and reasonably cheap) function $\text{inv}(\cdot)$ such that $a \otimes b \otimes \text{inv}(a) = b$ for all $a, b \in S$. In general, a monoid may not be commutative nor invertible.

In the context of aggregation, monoids strike a good balance between generality and efficiency as was demonstrated before [9, 34, 41]. For this reason, we focus our attention on supporting monoidal aggregation, formulating the abstract data type as follows:

Definition: The first-in first-out *sliding-window aggregation* (SWAG) abstract data type maintains a collection of window data and supports the following operations:

- *query()* returns the ordered monoidal product of the window data. That is, if the sliding window contains values v_0, v_1, \dots, v_{n-1} in their arrival order, *query* returns $v_0 \otimes v_1 \otimes \dots \otimes v_{n-1}$. If the window is empty, it returns $\mathbf{1}$.
- *insert(v)* adds v to the end of the sliding window. That is, if the sliding window contains values v_0, v_1, \dots, v_{n-1} in their arrival order, then *insert(v)* updates the collection to v'_0, v'_1, \dots, v'_n , where $v'_i = v_i$ for $i = 0, 1, \dots, n-1$ and $v'_n = v$.

- *evict()* removes the oldest item from the front of the sliding window. That is to say, if the sliding window contains values v_0, v_1, \dots, v_{n-1} in their arrival order, then *evict()* updates the collection to $v'_0, v'_1, \dots, v'_{n-2}$, where $v'_i = v_{i+1}$ for $i = 0, 1, 2, \dots, n-2$.

Throughout, n will denote the size of the current sliding window and v_0, v_1, \dots, v_{n-1} will denote the contents of the sliding window in their arrival order, where v_0 is the oldest element. SWAG itself is not a concrete algorithm; it is merely an abstract data type, defining a set of operations with their expected behavior. The algorithms introduced in this paper (including Two-Stacks, DABA, and DABA Lite) are all concrete instantiations for the SWAG abstract data type.

2.2 Aggregation on Monoids

Despite their simplicity, monoids are expressive enough to capture most basic aggregations [9, 34], as well as more sophisticated aggregations such as maintaining approximate membership via a Bloom filter [8], maintaining an approximate count of distinct elements [14], maintaining the versatile count-min sketch [12], and indeed all operators in Table 1.

However, many aggregations (e.g., standard deviation) are not themselves monoids but can be couched as operations on a monoid with the help of two extra steps. To accomplish this, prior work [34] gives a framework for the developer to provide three types *In*, *Agg*, and *Out*, and write three functions as follows:

- *lift(e : In) : Agg* takes an element of the input type and “lifts” it to an aggregation type that will be monoid operable.
- *combine(v₁ : Agg, v₂ : Agg) : Agg* is a binary operator operating on the aggregation type. In our paper’s terminology, *combine* is the monoidal binary operator \otimes .
- *lower(v : Agg) : Out* turns an element of the aggregation type into an element of the output type.

Consider the example of maintaining the *maxcount*, which yields the number of times the maximal value occurs in the window. Define the type *Agg* as a pair $\langle m, c \rangle$ comprising the maximum m and its count c . Then, define the three functions *lift*, *combine*, and *lower* as:

$$\begin{aligned} \text{lift}(e) &= \langle m \mapsto e, c \mapsto 1 \rangle \\ \text{combine}(v_1, v_2) &= \begin{cases} v_1 & \text{if } v_1.m > v_2.m \\ v_2 & \text{if } v_1.m < v_2.m \\ \langle m \mapsto v_1.m, \\ c \mapsto v_1.c + v_2.c \rangle & \text{if } v_1.m = v_2.m \end{cases} \\ \text{lower}(v) &= v.c \end{aligned}$$

It is easy to show that the *combine* function is an associative binary operator with identity element $\mathbf{1} = \langle -\infty, 0 \rangle$. Consequently, $\mathcal{M}_{\text{maxcount}} = (\text{Agg}, \text{combine}, \langle -\infty, 0 \rangle)$ is a monoid.

In this framework, a query is *conceptually* answered as follows. If the sliding window currently contains the elements e_0, e_1, \dots, e_{n-1} , from the earliest to the latest, then *lift* derives $v_i = \text{lift}(e_i)$ for $i = 0, 1, 2, \dots, n-1$. Then, *combine*, rendered as infix \otimes , computes $v = v_0 \otimes v_1 \otimes \dots \otimes v_{n-1}$. Finally, *lower* produces the final answer as $\text{lower}(v)$.

Note that *lift* only needs to be applied to each element when it first arrives and *lower* to query results at the end. Therefore, the present paper focuses exclusively on the issue of maintaining the monoidal product—i.e., how to make as few invocations of *combine* as possible.

2.3 Example Trace

Using the maxcount monoid mentioned previously as a running example for the following sections, we will look at a trace of window operations and their effects on aggregations. Consider a window with the following contents, with the oldest element on the left and the youngest on the right.

4, 5, 3, 4, 0, 4, 4, max=5, maxcount=1

The largest number in the window is 5, and it occurs only once, so the maxcount is 1. The oldest element on the left is 4, which is smaller than the current maximum 5, so evicting it does not affect the maximum or the maxcount.

5, 3, 4, 0, 4, 4, max=5, maxcount=1

If we again evict the oldest element from the left, the maximum remaining window element becomes 4. Since the number 4 occurs thrice in the window, the maxcount is 3. The monoid is not invertible: this update could not have been accomplished by “subtracting out” information from the previous partial aggregate.

3, 4, 0, 4, 4, max=4, maxcount=3

Inserting 2 does not affect the maximum, and hence, it also does not affect the maxcount.

3, 4, 0, 4, 4, 2, max=4, maxcount=3

Finally, inserting 6 changes the maximum. Since the newly inserted element is the only 6 in the window, the maxcount becomes 1.

3, 4, 0, 4, 4, 2, 6, max=6, maxcount=1

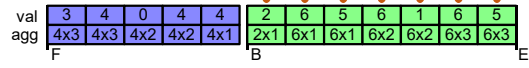
Notice that in this trace, *insert* and *evict* do not strictly alternate. In general, the SWAG data type, as well as all our algorithms, places no restrictions on how *insert* and *evict* may be called. They can be arbitrarily interleaved, allowing for dynamically-sized windows.

3 Two-Stacks

Two-Stacks is a simple algorithm for in-order sliding window aggregation (SWAG). For a window size n , it stores a total of $2n$ partial aggregates and implements each SWAG operation with amortized $O(1)$ and worst-case $O(n)$ invocations of \otimes .

The text of this section embeds several data-structure visualizations, which are all taken from concrete and complete example traces shown in Figs. 1 and 2.

Two-Stacks Data Structure. As the name implies, the data structure for Two-Stacks comprises two stacks. We refer to them as the front stack F and the back stack B . Here is an example of these two stacks for maxcount aggregation:



The front stack is shown in blue rotated 90° left, with its top marked F and its bottom marked B . The back stack is shown in green rotated 90° right, with its bottom marked B and its top marked E . To avoid clutter, the visualization only shows B once to mark the bottoms of both the front stack and the back stack. As the window slides, evictions pop elements from the front stack on the left (at F) and insertions push elements on the back stack on the right (at E). Each stack element is a struct with two partial aggregates, *val* shown on top and *agg* shown on the bottom. The notation 4×3 is shorthand for max=4, maxcount=3.

For amortized analysis, we use the banker’s (aka. accounting) method [11], which keeps an imaginary savings account. In this method, every operation is amortized $O(1)$ if we can show that by billing the user a constant amount for every operation invoked, there is enough money at all time, without taking out a loan, to pay for the the actual work being done. We visualize the savings as small golden “coins” above the elements; they are not actual manifest in the data structure.

Two-Stacks Invariants. An invariant for a data structure that implements in-order SWAG is a property that holds before and after every *query*, *insert*, or *evict*. Let v_0, \dots, v_{n-1} be the lifted partial aggregates of the current window contents. Each *val* field stores the corresponding v_i . Each *agg* field stores the partial aggregate of the corresponding v_i and all other values below it in the same stack. Formally, if $F[i]$ and $B[i]$ denote the i^{th} element of F and B indexed from the left starting at index 0:

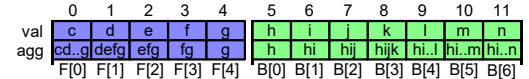
$$\forall i \in 0 \dots |F| - 1 : F[i].\text{val} = v_i$$

$$\text{and } \forall i \in 0 \dots |F| - 1 : F[i].\text{agg} = v_i \otimes \dots \otimes v_{|F|-1}$$

$$\text{and } \forall i \in |F| \dots |F| + |B| - 1 : B[i - |F|].\text{val} = v_i$$

$$\text{and } \forall i \in |F| \dots |F| + |B| - 1 : B[i - |F|].\text{agg} = v_{|F|} \otimes \dots \otimes v_i$$

The front stack aggregates to the right (easy eviction from the left). The back stack aggregates to the left (easy insertion from the right). Here is a visual example of the invariants:



The notation $cd.g$ is shorthand for $c \otimes d \otimes \dots \otimes g$. To work correctly regardless of commutativity, the aggregation in both stacks is ordered from the older elements on the left to newer elements on the right. For example, we take care to aggregate $f \otimes g$ instead of $g \otimes f$ because f is older than g .

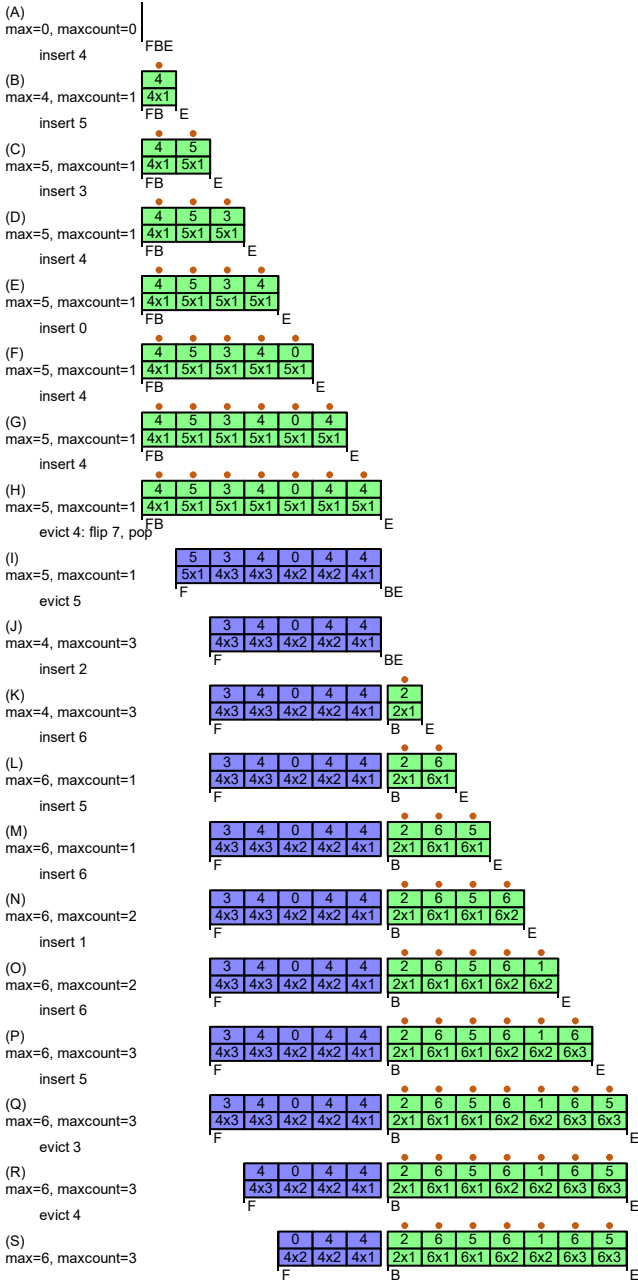


Fig. 1 Two-Stacks example trace for maxcount aggregation. The notation $m \times c$ is shorthand for $\max=m$, $\maxcount=c$.

Two-Stacks Algorithm. Being an algorithm that implements in-order SWAG, Two-Stacks needs to define the functions *query*, *insert*, and *evict*. But first, we will define two private helper functions that retrieve the partial aggregate of the entire front stack *F* and back stack *B*, respectively.

```

1 fun  $\Pi_F^\otimes$ 
2   if F.isEmpty() return 1 else return F.top().agg
3 fun  $\Pi_B^\otimes$ 
4   if B.isEmpty() return 1 else return B.top().agg

```

Recall that 1 is the identity element of the monoid. These helpers return the correct values in constant time, thanks to

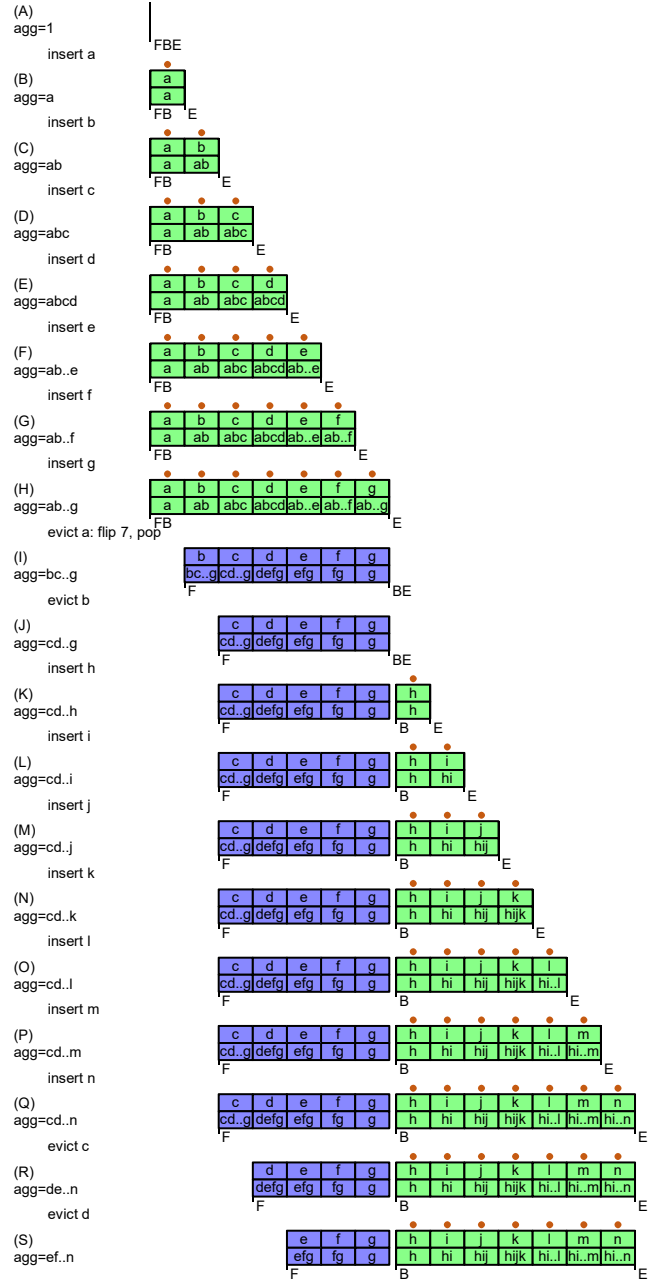


Fig. 2 Two-Stacks example trace for any aggregation. The notation for aggregates omits \otimes , e.g., $bc..g$ is shorthand for $b \otimes c \otimes \dots \otimes g$.

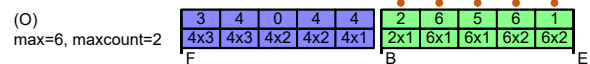
the invariants discussed previously. Function *query* combines the results of both helpers, using a single invocation of \otimes .

```

5 fun query()
6   return  $\Pi_F^\otimes \otimes \Pi_B^\otimes$ 

```

As an example, given the following data structure state, Π_F^\otimes is 4×3 and Π_B^\otimes is 6×2 , so *query* returns the maximum 6×2 .



Next, to define *insert* and *evict*, we can assume that the invariants hold before function calls and must guarantee that

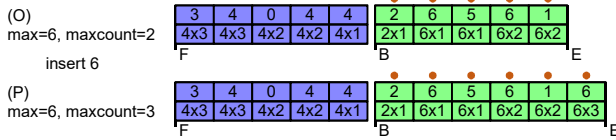
they hold afterwards. The *insert* function just pushes onto B , taking constant time. Assuming the invariants hold for the old top of B , *insert* guarantees the invariants for the new top of B by setting its partial aggregate *agg* to $\Pi_B^\otimes \otimes v$.

```

7 fun insert(v)
8   B.push(v,  $\Pi_B^\otimes \otimes v$ )

```

The following example illustrates *insert*.



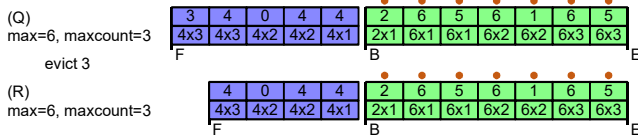
Finally, *evict* pops from F after first ensuring that F is nonempty.

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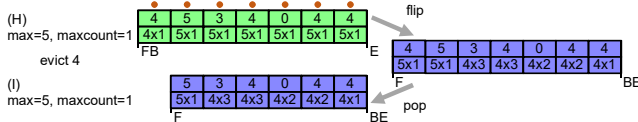
9 fun evict()
10  if F.isEmpty() # Flip
11    while not B.isEmpty()
12      F.push(B.top().val, B.top().val  $\otimes \Pi_F^\otimes$ )
13      B.pop()
14  F.pop()

```

If F is nonempty, then *evict* is trivial, for example:



On the other hand, if F is empty, then *evict* must first do a *flip*. The *flip* operation pushes all values from B onto F and reverses the direction of the aggregation. In other words, it establishes that the *agg* fields satisfy the invariant for F . After the flip, *evict* simply does a *pop* as before.



Because of the reversal loop, *flip* takes $O(n)$ time, where n is the current number of elements. Nevertheless, *evict* only takes amortized $O(1)$ time, as we will see below.

Two-Stacks Theorems.

Lemma 1 *Two-Stacks maintains the invariants listed above.*

Proof The *query* function does not modify the stacks and thus does not change the invariants. The *insert* function maintains the invariants by correctly setting *agg* for the newly pushed element. The *evict* function maintains the invariants by correctly setting *agg* for all elements of F during flip. \square

Theorem 2 *If the window currently contains v_0, \dots, v_{n-1} , then *query* returns $v_0 \otimes \dots \otimes v_{n-1}$.*

Proof Using Lemma 1,

$$\begin{aligned}
 & \text{query}() \\
 &= \Pi_F^\otimes \otimes \Pi_B^\otimes \\
 &= v_0 \otimes \dots \otimes v_{|F|-1} \otimes v_{|F|} \otimes \dots \otimes v_{|F|+|B|-1} \\
 &= v_0 \otimes \dots \otimes v_{n-1}
 \end{aligned}$$

\square

Theorem 3 *Two-Stacks requires space to store $2n$ partial aggregates. Each call to *query* and *insert* invokes \otimes exactly one time. Each call to *evict* invokes \otimes at most n times and amortized 1 time.*

Proof The only part of the theorem that is not immediately obvious is the amortized complexity of *evict*. To see this, we bill each call to *insert* two imaginary coins: one for pushing an element onto B and one to go into the savings. Hence, every element in B , as visualized, has a golden coin on top of it. When *flip* happens, it invokes \otimes once for every element of B , which is completely paid for by spending the coin on that element. Because billing a constant amount per operation covers the total cost, each operation is amortized $O(1)$. \square

To summarize the workings of Two-Stacks, we will have another look at Figs. 1 and 2, which show complete example traces. Insertions push to the right of the back stack, visualized in green. Evictions pop from the left of the front stack, visualized in blue. Given an empty front stack, *evict* first performs a flip, as shown in Step (H) \rightarrow (I). The flip keeps values unchanged but reverses the associated partial aggregates. In this example, there are 7 partial aggregates to flip, coming from the preceding 7 *insert* operations, which have deposited 7 coins to the savings to pay for the flip.

4 Two-Stacks Lite

Two-Stacks Lite improves upon Two-Stacks by reducing the space complexity from $2n$ down to $n + 1$ stored partial aggregates. It does this by exploiting the insight that the Two-Stacks algorithm reads none of the *val* fields of the front stack and reads only the last *agg* field of the back stack. This idea comes from the Hammer Slide paper by Theodorakis et al. [35]. Another improvement is that instead of physically maintaining two separate stacks, Two-Stacks Lite maintains a single double-ended queue with an internal pointer B to track the virtual stack boundary. The time complexity is unchanged, at amortized $O(1)$ and worst-case $O(n)$ invocations of \otimes per SWAG operation. The data-structure visualizations in this section are taken from the concrete example traces shown in Figs. 3 and 4.

Two-Stacks Lite Data Structure. The data structure for Two-Stacks Lite comprises a double-ended queue *deque* of partial aggregates, one additional partial aggregate *aggB*, and three pointers F , B , and E . Pointer F points to the start of *deque*, B points to a location between start and end, and E points to

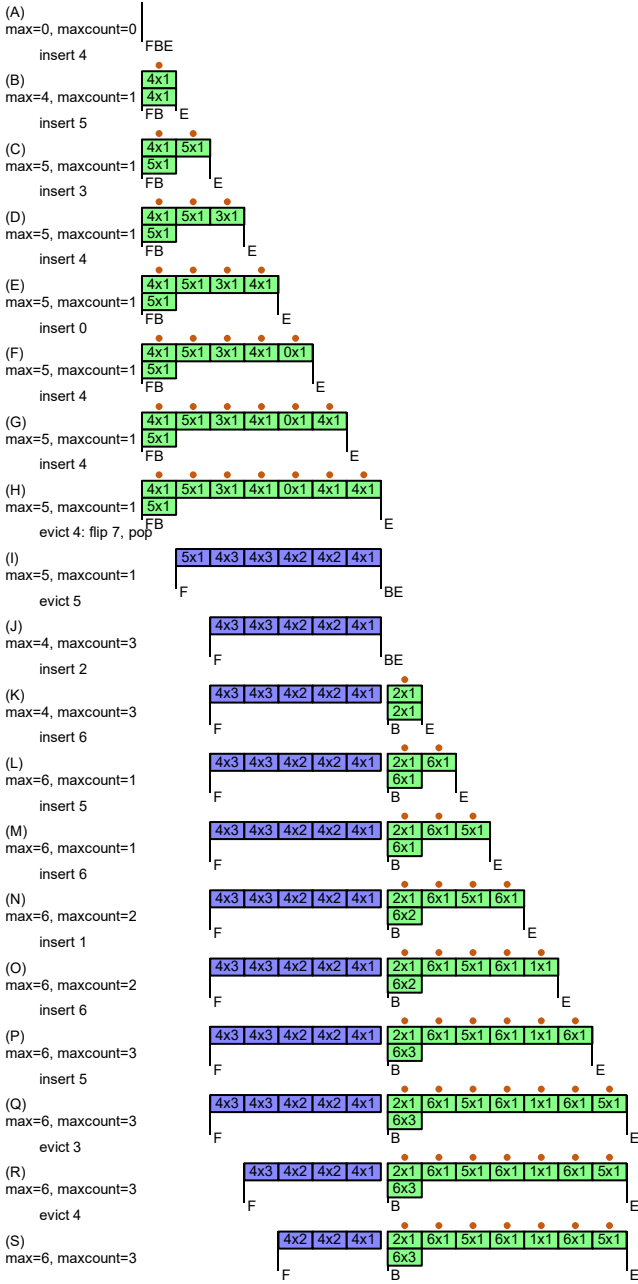
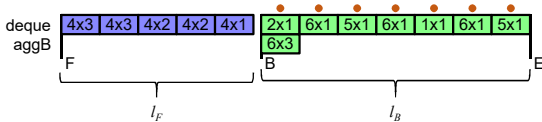


Fig. 3 Two-Stacks Lite example trace for maxcount aggregation. The notation $m \times c$ is shorthand for $\text{max}=m, \text{maxcount}=c$.

the end. Here is an example with a max-count aggregation (the golden “coins” visualizing the savings serve the same purpose as in Two-Stacks):



Definition 4 (Pointers) In this paper, a *pointer* is an iterator into a resizable double-ended queue that supports the following basic data structure operations:

- dereference and read or write contents: $v \leftarrow *p, *p \leftarrow v$

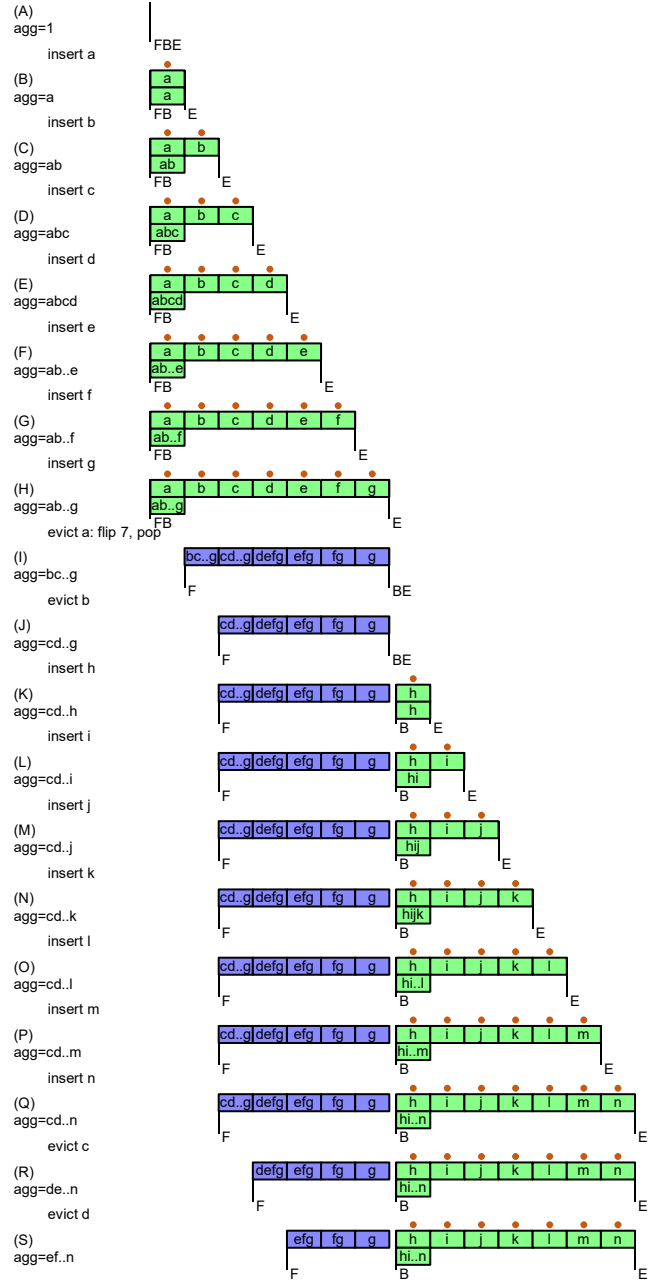


Fig. 4 Two-Stacks Lite example trace for any aggregation. The notation for aggregates omits \otimes , e.g., $bc..f$ is shorthand for $b \otimes c \otimes .. \otimes f$.

- pointer comparison: $p = q, p \neq q$
- pointer increment and decrement: $p + 1, p - 1$
- pointer assignment: $p \leftarrow q$

Our algorithms only use the above-listed pointer operations. These pointer operations can be implemented in worst-case $O(1)$ time over a variable-sized double-ended queue by implementing that queue using chunked arrays [31]. For stating invariants, we will also use a few additional pointer operations such as $p + i$, $p - q$, or $p < q$. These additional operations do not need to be $O(1)$ since they are used only by invariants and not directly by our algorithms. \square

Though physically the data structure uses a single deque, the pointers demarcate two virtual sublists, l_F and l_B .

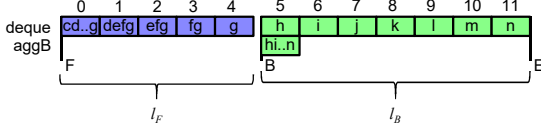
Two-Stacks Lite Invariants. Let v_0, \dots, v_{n-1} be the current window contents from the oldest to the youngest. Then, each deque element in the front sublist l_F ($F \leq p < B$) stores the partial aggregate starting from the corresponding v_i up to the element before B . Each deque element in the back sublist l_B ($B \leq p < E$) stores the corresponding v_i . In addition, $aggB$ stores the partial aggregate of all elements in l_B . Formally:

$$\forall i \in 0 \dots B - F - 1 : *(F + i) = v_i \otimes \dots \otimes v_{B-F-1}$$

$$\text{and } \forall i \in B - F \dots E - F - 1 : *(F + i) = v_i$$

$$\text{and } aggB = *B \otimes \dots \otimes *(E - 1)$$

As an example, assume that the current window contains the values $v_0 = c, v_1 = d, v_2 = e$, etc. up to $v_{10} = m$ and $v_{11} = n$. Then, if there are five elements in l_F and seven in l_B , the data structure looks as follows:



The notation $cd..g$ is shorthand for $c \otimes d \otimes \dots \otimes g$.

Two-Stacks Lite Algorithm. The private helper function Π_F^\otimes retrieves the partial aggregate of the entire front sublist l_F , where $\mathbf{1}$ is the identity element of the monoid.

```

1 fun  $\Pi_F^\otimes$ 
2   if ( $F = B$ ) return  $\mathbf{1}$  else return  $*F$ 

```

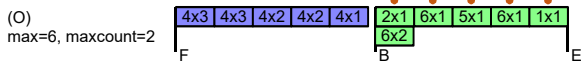
Function *query* obtains the partial aggregates of l_F (by calling Π_F^\otimes) and of l_B (by reading $aggB$) and combines them in constant time.

```

3 fun query()
4   return  $\Pi_F^\otimes \otimes aggB$ 

```

As an example, given the following data structure state, Π_F^\otimes is 4×3 and $aggB$ is 6×2 , so *query* returns the maximum 6×2 .



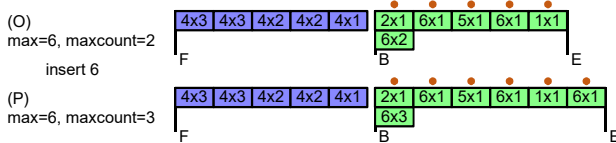
Both *insert* and *evict* can assume that the invariants hold before they are called and must guarantee that they hold afterwards. Function *insert*(v) accomplishes this by pushing v and updating $aggB$ accordingly.

```

5 fun insert(v)
6   deque.pushBack(v)
7    $E \leftarrow E + 1$ 
8    $aggB \leftarrow aggB \otimes v$ 

```

The following example illustrates *insert*.



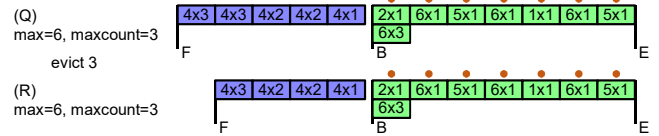
Finally, *evict* pops from the front, after first ensuring that the front sublist l_F is nonempty.

```

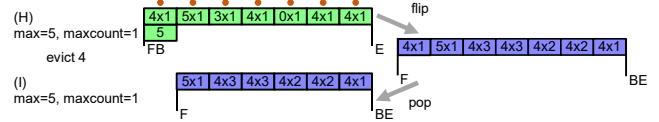
9 fun evict()
10  if  $F = B$  and  $B \neq E$  # Flip
11     $I \leftarrow E - 1$ 
12    while  $I \neq F$ 
13       $I \leftarrow I - 1$ 
14       $*I \leftarrow *I \otimes *(I + 1)$ 
15     $B \leftarrow E$ 
16     $aggB \leftarrow \mathbf{1}$ 
17     $F \leftarrow F + 1$ 
18    deque.popFront()

```

If l_F is nonempty, then *evict* is trivial, for example:



If l_F is empty, then *evict* does a *flip*. Lines 11–14 rewrite the contents of *deque* in-place to contain partial aggregates from the corresponding element to the end. Line 15 updates pointer B to indicate that the front sublist l_F now occupies the entire data structure and the back sublist l_B is empty. Then, Line 16 resets $aggB$ to the monoid identity element. Here is a visualization of flip followed by pop:



The loop takes time $O(n)$ but can be amortized over the preceding n insertions, where n is the current window size.

Two-Stacks Lite Theorems.

Lemma 5 *The Two-Stacks Lite algorithm maintains the Two-Stacks Lite invariants.*

Proof By inspection, function *query* preserves the data and thus the invariants. Function *insert* maintains the invariants by pushing v and updating $aggB$. Function *evict* optionally does a flip, which reestablishes the invariants, then always does a pop, which also maintains the invariants. \square

Theorem 6 *If the window currently contains v_0, \dots, v_{n-1} , then *query* returns $v_0 \otimes \dots \otimes v_{n-1}$.*

Proof Using Lemma 5,

$$\begin{aligned}
 \text{query}() &= \Pi_F^\otimes \otimes aggB \\
 &= v_0 \otimes \dots \otimes v_{B-F-1} \otimes v_{B-F} \otimes \dots \otimes v_{E-F-1} \\
 &= v_0 \otimes \dots \otimes v_{n-1}
 \end{aligned}$$

\square

Theorem 7 *Two-Stacks Lite requires space to store $n + 1$ partial aggregates. Each call to *query* and *insert* invokes \otimes exactly one time. Each call to *evict* invokes \otimes at most n times and amortized one time.*

Proof Most of the theorem is obvious. To prove the amortized complexity of *evict*, bill each call to *insert* two imaginary coins for pushing an element onto the back and for the savings (visualized as small golden “coin” above the elements). Hence, every element in l_B has a golden coin on top of it. When *flip* happens, it invokes \otimes once for every element of l_B , which is completely paid for by spending the coin on that element. \square

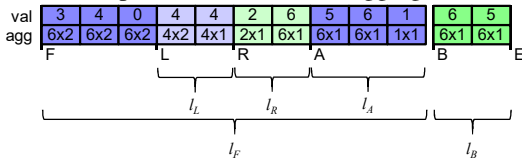
5 DABA

DABA is a low-latency algorithm for in-order sliding window aggregation (SWAG). When the window has size n , DABA requires space to store $2n$ partial aggregates and supports each SWAG operation using worst-case $O(1)$ invocations of \otimes . For a brief explanation of the name, DABA stands for De-Amortized Banker’s Aggregator: Amortization looks at the average cost of an operation over a long period of time. The banker’s method conceptualizes amortization as moving imaginary coins between the algorithm and a fictitious bank. Deamortization is a method that turns the average-case behavior into the worst-case behavior, usually by carefully spreading out expensive operations. In this spirit, notice that the expensive operation in the Two-Stacks algorithm is the loop for reversing the direction of aggregation during flip, paid for by imaginary coins deposited on preceding insertions. Whereas Two-Stacks does the flip late when the front stack becomes empty, DABA does the flip earlier, when the front and back stack reach the same length. Furthermore, instead of doing a reversal loop at the time of the flip, DABA spreads out the steps for reversing the direction of aggregation. The text of this section embeds several data-structure visualizations taken from concrete example traces shown in Figs. 5 and 6.

DABA Data Structure. The DABA data structure comprises a double-ended queue, *deque*, and six pointers, F, L, R, A, B , and E , into that queue. Each queue element is a struct with two partial aggregates: *val* (top row) and *agg* (bottom row). The basic pointer operations are the same as in Definition 4 and are easy to implement in $O(1)$ time. The pointers are always ordered as follows:

$$F \leq L \leq R \leq A \leq B \leq E$$

Here is an example with a max-count aggregation:



The *val* fields store the window contents, with the i^{th} oldest value in FIFO order stored at $v_i = *(F + i).val$. The *agg* fields store partial aggregations over subranges of the window. Conceptually, each pointer p corresponds to a sublist l_p .

For example, pointer F corresponds to sublist l_F . Each sublist is either aggregated to the left or to the right. The direction is carefully chosen to enable the DABA operations. The left-most portion of the front list l_F is aggregated to the left to facilitate eviction. The back list l_B is aggregated to the right to facilitate insertion. The inner sublists l_L, l_R , and l_A are designed to facilitate incremental reversal. Incremental reversal happens by adjusting the pointers demarcating sublist boundaries one step at a time. When a pointer moves, a deque element changes membership from one sublist to another and its *agg* field may need to be updated accordingly.

DABA Invariants. DABA maintains three groups of invariants: values invariants, partial aggregate invariants, and size invariants. DABA’s *values invariants* specify that the *val* field of each element stores a singleton partial aggregate v_i obtained by lifting the corresponding single stream element.

$$\forall i \in 0 \dots E - F - 1 : *(F + i).val = v_i$$

DABA’s *partial aggregate invariants* specify the contents of the *agg* fields before and after each SWAG operation, based on sublists. In the visualizations, blue indicates aggregation to the right and green indicates aggregation to the left. In the left-most portion of sublist l_F (the front sublist, in dark blue), each *agg* field holds an aggregate starting from that element to the right end of l_F . In sublist l_L (the left sublist, in light blue), each *agg* field holds an aggregate starting from that element to the right end of l_L . In sublist l_R (the right sublist, in light green), each *agg* field holds an aggregate starting from the left end of l_R to that element. In sublist l_A (the accumulator sublist, in dark blue), each *agg* field holds an aggregate starting from that element to the right end of l_A , which coincides with the right end of l_F . Finally, in sublist l_B (the back sublist, in dark green), each *agg* field holds an aggregate starting from the left end of l_B to that element. Formally:

$$\forall i \in 0 \dots L - F - 1 : *(F + i).agg = v_i \otimes \dots \otimes v_{B-F-1}$$

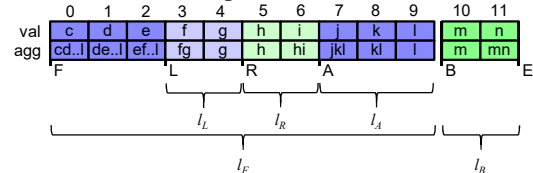
$$\text{and } \forall i \in L - F \dots R - F - 1 : *(F + i).agg = v_i \otimes \dots \otimes v_{R-F-1}$$

$$\text{and } \forall i \in R - F \dots A - F - 1 : *(F + i).agg = v_{R-F} \otimes \dots \otimes v_i$$

$$\text{and } \forall i \in A - F \dots B - F - 1 : *(F + i).agg = v_i \otimes \dots \otimes v_{B-F-1}$$

$$\text{and } \forall i \in B - F \dots E - F - 1 : *(F + i).agg = v_{B-F} \otimes \dots \otimes v_i$$

Here is a visual example of the invariants:



The notation $cd..l$ is shorthand for $c \otimes d \otimes \dots \otimes l$. To work correctly irrespective of whether the monoid is commutative or not, in all sublists, the operands of \otimes are always ordered from older on the left to newer on the right.

DABA’s *size invariants* specify constraints on the sizes of sublists. Given a pointer p , we use the notation $|l_p|$ to indicate the size of sublist l_p . For example, the size of sublist

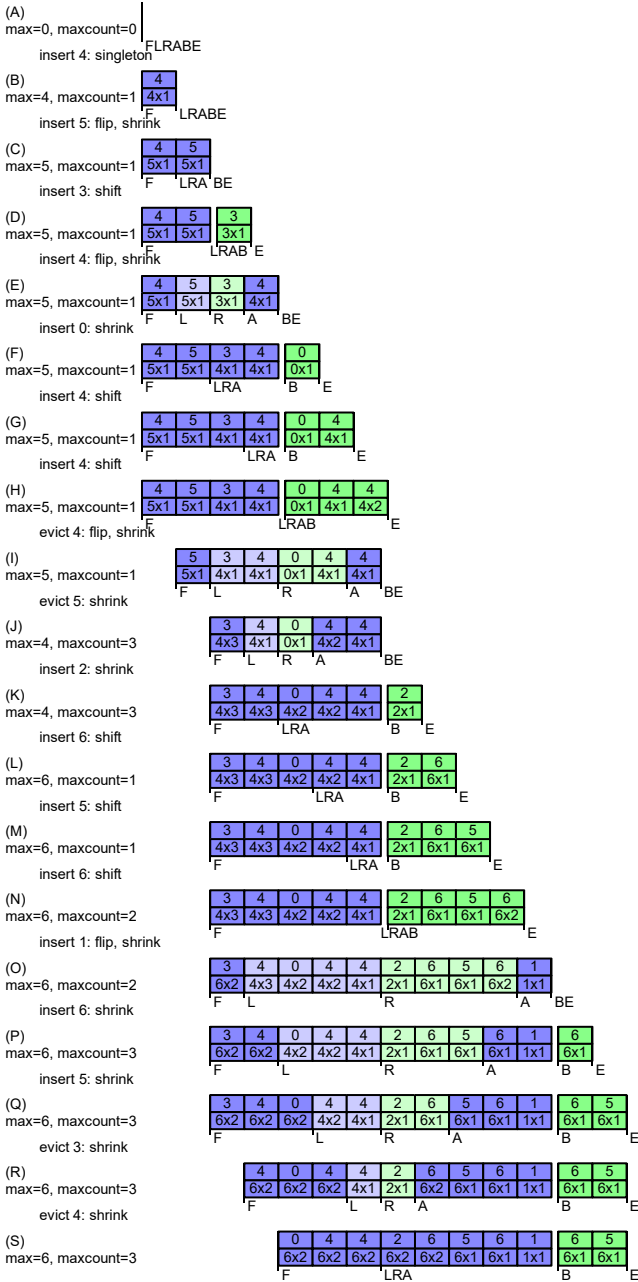


Fig. 5 DABA example trace for maxcount aggregation. The notation $m \times c$ is shorthand for $\max=c, \maxcount=m$.

l_F is $|l_F| = B - F$ and the size of sublist l_L is $|l_L| = R - L$. Formally, the size invariants are

$$(|l_F| = 0 \wedge |l_B| = 0) \vee (|l_L| + |l_R| + |l_A| + 1 = |l_F| - |l_B| \wedge |l_L| = |l_R|)$$

This says that the window is either empty ($|l_F| = 0$ and $|l_B| = 0$) or the following two conditions hold:

- First, $|l_L| + |l_R| + |l_A| + 1 = |l_F| - |l_B|$. The size of the front list l_F exceeds the size of the back list l_B by the total size of the sublists l_L , l_R , and l_A plus one. The sublists l_L , l_R , and l_A are used for incremental reversal, and the algorithm, shown below, shrinks their total size by one on

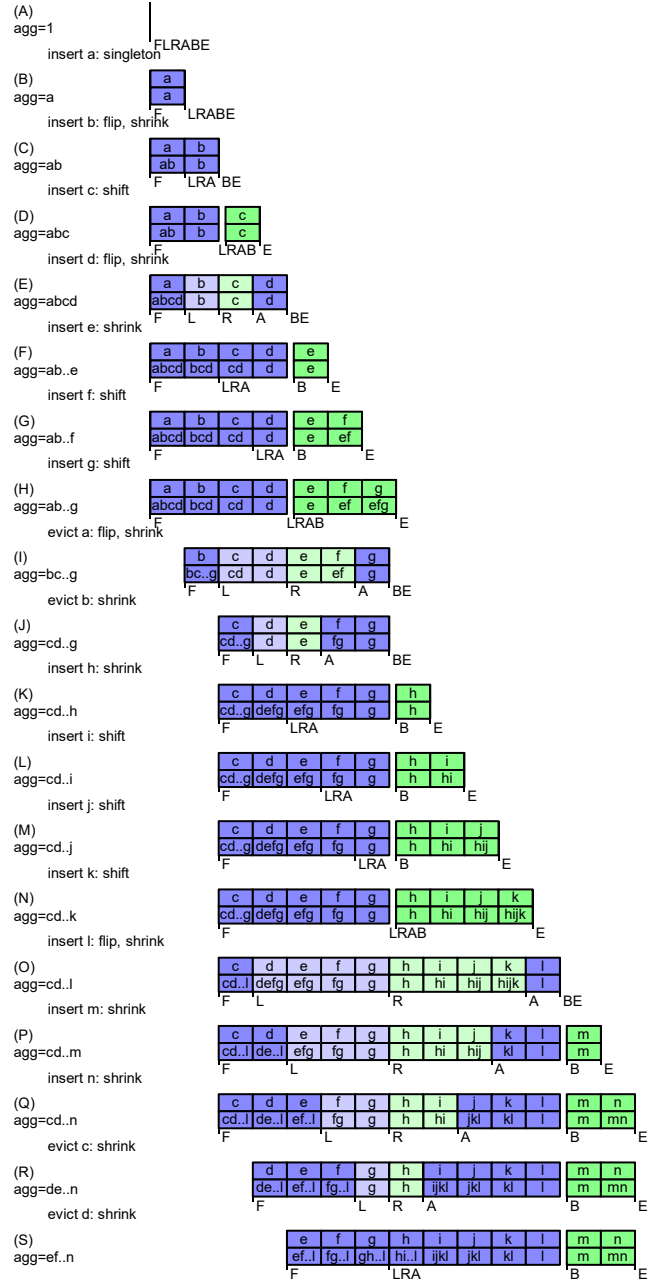


Fig. 6 DABA example trace for any aggregation. The notation for aggregates omits \otimes , e.g., $bc..f$ is shorthand for $b \otimes c \otimes .. \otimes f$.

each insertion or eviction. When the sublists l_L , l_R , and l_A are empty, the algorithm can make just one more insertion or eviction before l_F and l_B reach the same size. At that point, the algorithm does a *flip*, which relabels l_F and l_B into l_L and l_R , respectively.

- Second, $|l_L| = |l_R|$. After each *flip*, l_L and l_R start out with the same size and then shrink at the same pace.

Below, we will see explanations for how the algorithm maintains these invariants, using color-coding to recognize corresponding subequations.

DABA Algorithm. For each sublist l_p , a private helper function Π_p^\otimes retrieves the corresponding partial aggregate or returns the monoid's identity element $\mathbf{1}$ if the sublist is empty. Note that for a given sublist, we retrieve the partial aggregate in the left-most element if that sublist aggregates to the right, and the partial aggregate in the right-most element if that sublist aggregates to the left.

```

1 fun  $\Pi_F^\otimes$ 
2   if  $(F = B)$  return  $\mathbf{1}$  else return  $*F.agg$ 
3 fun  $\Pi_B^\otimes$ 
4   if  $(B = E)$  return  $\mathbf{1}$  else return  $*(E-1).agg$ 
5 fun  $\Pi_L^\otimes$ 
6   if  $(L = R)$  return  $\mathbf{1}$  else return  $*L.agg$ 
7 fun  $\Pi_R^\otimes$ 
8   if  $(R = A)$  return  $\mathbf{1}$  else return  $*(A-1).agg$ 
9 fun  $\Pi_A^\otimes$ 
10  if  $(A = B)$  return  $\mathbf{1}$  else return  $*A.agg$ 

```

These helpers return the correct values in constant time, thanks to the invariants defined previously. Function *query* combines the aggregate of l_F and l_B , taking only a single invocation of \otimes .

```

11 fun query()
12   return  $\Pi_F^\otimes \otimes \Pi_B^\otimes$ 

```

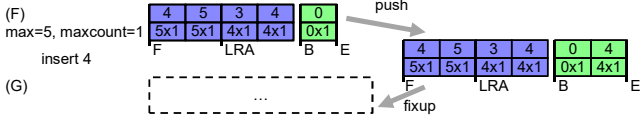
Function *insert* pushes a value v with corresponding partial aggregate to the back of the deque, then calls a function *fixup*, defined below, for doing one step of incremental reversal.

```

13 fun insert(v)
14   deque.pushBack(v,  $\Pi_B^\otimes \otimes v$ )
15    $E \leftarrow E + 1$ 
16   fixup()

```

In our running maxcount example, if $\Pi_B^\otimes = 0$ and $v = 4$, then the newly pushed deque element has $val = 4$ and $agg = 4$.



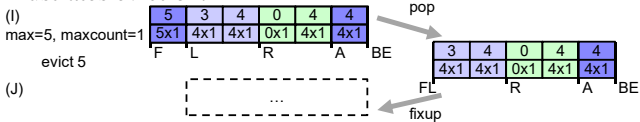
Similarly, *evict* pops an element from the front of the deque, then calls *fixup* for one step of incremental reversal.

```

17 fun evict()
18    $F \leftarrow F + 1$ 
19   deque.popFront()
20   fixup()

```

In our running maxcount example, the following picture illustrates eviction:



The *fixup* function is responsible for restoring the invariants. Recall that we can assume that the following size invariants hold before each call to *insert* or *evict*:

$$(|l_F| = 0 \wedge |l_B| = 0) \vee (|l_L| + |l_R| + |l_A| + 1 = |l_F| - |l_B| \wedge |l_L| = |l_R|)$$

Function *insert* grows l_B by one element and function *evict* shrinks l_F by one element. The impact on the invariants is

the same in both cases: they both decrease the difference $|l_F| - |l_B|$ by one. Thanks to the extra $+1$ element in l_F , neither *insert* nor *evict* affects the inner sublists l_L , l_R , and l_A . This means that upon entry to *fixup*, the following is true:

$$(|l_F| = 0 \wedge |l_B| = 1) \vee (|l_L| + |l_R| + |l_A| = |l_F| - |l_B| \wedge |l_L| = |l_R|)$$

Using the above as a precondition, the postcondition of *fixup* is to reestablish the original size invariants. The *fixup* function does this via four cases *singleton*, *flip*, *shift*, and *shrink*.

```

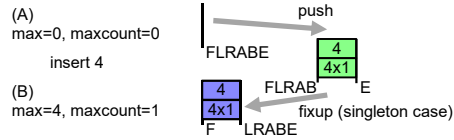
21 fun fixup()
22   if  $F = B$  # Singleton case
23      $B \leftarrow E, A \leftarrow E, R \leftarrow E, L \leftarrow E$ 
24   else
25     if  $L = B$  # Flip
26        $L \leftarrow F, A \leftarrow E, B \leftarrow E$ 
27     if  $L = R$  # Shift
28        $A \leftarrow A + 1, R \leftarrow R + 1, L \leftarrow L + 1$ 
29     else # Shrink
30        $*L.agg \leftarrow \Pi_L^\otimes \otimes \Pi_R^\otimes \otimes \Pi_A^\otimes$ 
31        $L \leftarrow L + 1$ 
32        $*(A-1).agg \leftarrow *(A-1).val \otimes \Pi_A^\otimes$ 
33        $A \leftarrow A - 1$ 

```

The *singleton* case happens when $|l_F| = 0$. Given the precondition, this can only hold when $|l_B| = 1$. Then, without having to modify the deque, the pointer assignments

$$B \leftarrow E, A \leftarrow E, R \leftarrow E, L \leftarrow E$$

change the size of l_F to $|l_F| = 1$ while making all the other sublists l_L , l_R , l_A , and l_B empty, as illustrated below.



The *singleton* case code thus establishes

$$(|l_L| + |l_R| + |l_A| + 1 = |l_F| - |l_B| = 1 \wedge |l_L| = |l_R| = 0)$$

which implies the original size invariants.

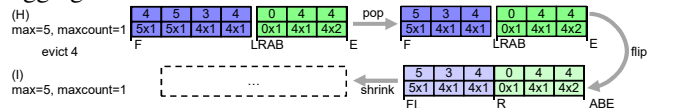
The *flip* case happens when $|l_F| > 0$ and the sublists for incremental reversal are empty: $|l_L| + |l_R| + |l_A| = 0$. Together with the precondition, this implies that

$$(|l_L| + |l_R| + |l_A| = 0 \wedge |l_F| = |l_B| \wedge |l_L| = 0 \wedge |l_R| = 0 \wedge |l_A| = 0)$$

Then, the pointer assignments

$$L \leftarrow F, A \leftarrow E, B \leftarrow E$$

turn the old outer sublists l_F and l_B into the new inner sublists l_L and l_R , respectively. No updates to *agg* fields are required because the corresponding sublists already have the correct aggregation direction.



After flip, the following holds:

$$(|l_L| + |l_R| + |l_A| = |l_F| \wedge |l_B| = 0 \wedge |l_L| = |l_R| > 0)$$

At this point, we still need to execute the shrink case to repair the size invariants.

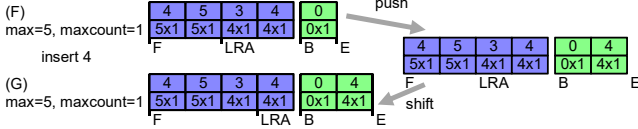
The *shift* case happens when $|l_F| > 0$ and $|l_L| = 0$. Together with the precondition, this implies that

$$(|l_L| + |l_R| + |l_A| = |l_F| - |l_B| \wedge |l_L| = 0 \wedge |l_R| = 0 \wedge |l_A| > 0)$$

Then, the pointer assignments

$$A \leftarrow A + 1, R \leftarrow R + 1, L \leftarrow L + 1$$

increment the pointers separating the left-most portion of l_F from l_A by one. No updates to *agg* fields are required because both the left-most portion of l_F and l_A are governed by the same *aggs* invariants.



After shift, the following holds:

$$(|l_L| + |l_R| + |l_A| + 1 = |l_F| - |l_B| \wedge |l_L| = 0 \wedge |l_R| = 0)$$

which implies the original size invariants.

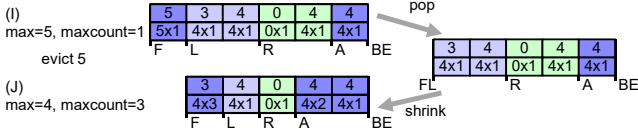
The *shrink* case happens when $|l_F| > 0$ and $|l_L| > 0$. There are two scenarios: with or without a flip from the same *fixup*. Either way, shrink starts with the following precondition:

$$(|l_L| + |l_R| + |l_A| = |l_F| - |l_B| \wedge |l_L| = |l_R| \wedge |l_L| > 0)$$

The shrink case is the only part of *fixup* that modifies not just pointers but also *agg* fields. It reduces the sizes of both l_L and l_R by one each. The top element of l_L becomes part of the left-most portion of l_F , so its *agg* field must be updated to $v_{L-F} \otimes \dots \otimes v_{B-F} = \Pi_L^\otimes \otimes \Pi_R^\otimes \otimes \Pi_A^\otimes$. The top element of l_R becomes part of the accumulator sublist l_A , so its *agg* field must be updated to $v_{A-F-1} \otimes \dots \otimes v_{B-F} = v_{A-F-1} \otimes \Pi_A^\otimes$.

$$\begin{aligned} *L.agg &\leftarrow \Pi_L^\otimes \otimes \Pi_R^\otimes \otimes \Pi_A^\otimes \\ L &\leftarrow L + 1 \\ *(A-1).agg &\leftarrow *(A-1).val \otimes \Pi_A^\otimes \\ A &\leftarrow A - 1 \end{aligned}$$

The following is a typical example of shrink, reducing the sizes of l_L and l_R from 2 to 1.



The following is an example of flip followed by shrink. After the flip, l_L and l_R both have size 3. After the shrink, l_L and l_R both have size 2.



Given the precondition of shrink, it establishes the following postcondition:

$$(|l_L| + |l_R| + |l_A| + 1 = |l_F| - |l_B| \wedge |l_L| = |l_R|)$$

which implies the original size invariants.

DABA Intuitive View. Now that we have seen all the small steps that make up DABA, let us look at their interplay to

help understand the algorithm more holistically. Figs. 5 and 6 show two variants of the same example trace, differing only in their aggregation monoid. The sequence of cases starting at (D) comprises [flip shrink shrink shift shift]. A similar pattern starts at (H), comprising [flip shrink shrink shrink shift shift shift]. More generally, each flip is followed by an equal number of shrink^m and shift^m cases. For instance, the sequence starting at (B) is [flip shrink shift], which is a special case where $m = 1$. Visually, during the shrink^m phase of the algorithm, the light blue and light green sublists narrow to a point, looking like an upside-down step pyramid. This corresponds to the incremental reversal of l_R , which of course was l_B before the flip. During the shrink^m phase, pointer R does not change. Afterwards, during the shift^m phase, the LRA pointers (which are now all the same) shift to the right one element at a time until they hit pointer B . When they reach B , the next insert or evict would cause l_F and l_B to have the same length, triggering the next flip and thus the next cycle. Within each cycle, pointer B always stays the same; it only moves when a flip happens.

DABA Theorems.

Lemma 8 *DABA maintains the invariants listed above, including the values invariants, the partial aggregate invariants, and the size invariants.*

Proof The query function does not modify the data structure and thus does not change the invariants. Functions *insert* and *evict* both establish the same precondition for *fixup*, as stated above. Finally, given that precondition, all cases of *fixup* reestablish the original invariants as a postcondition, as shown above. \square

Theorem 9 *If the window currently contains v_0, \dots, v_{n-1} , then query returns $v_0 \otimes \dots \otimes v_{n-1}$.*

Proof Using Lemma 8,

$$\begin{aligned} \text{query}() &= \Pi_F^\otimes \otimes \Pi_B^\otimes \\ &= v_0 \otimes \dots \otimes v_{B-F-1} \otimes v_{B-F} \otimes \dots \otimes v_{E-F-1} \\ &= v_0 \otimes \dots \otimes v_{n-1} \end{aligned}$$

\square

Theorem 10 *DABA requires space to store $2n$ partial aggregates. DABA invokes \otimes at most one time per query, four times per insert, and three times per evict. Furthermore, for nonempty windows, DABA invokes \otimes on average 2.5 times per insert and 1.5 times per evict.*

Proof The worst-case numbers can be seen directly from the code and by noting that the algorithm contains no loops or recursion. To see the average-case numbers, consider the sequence of *fixup* cases from a flip to the next. Immediately following flip, l_R is nonempty and l_A is empty. As long as

l_R is nonempty, each subsequent *insert* or *evict* executes a shrink, invoking \otimes three times. When l_R becomes empty, l_A has exactly the size that l_R had at the previous flip. As long as l_A is nonempty, each subsequent *insert* or *evict* executes a shift, without invoking \otimes . The next flip happens when l_A is empty. That means that there was an equal number of shrink steps as shift steps, and thus, an equal number of *fixup* calls with three invocations of \otimes and with zero invocations of \otimes . This averages out to 1.5 \otimes -invocations per *fixup*, and thus, 2.5 \otimes -invocations per *insert* and 1.5 per *evict*. \square

A corollary of Theorem 10 is that DABA implements all SWAG operations with worst-case $O(1)$ invocations of \otimes . Unlike previous algorithms in the paper, DABA involves no costly steps that would require an amortization argument.

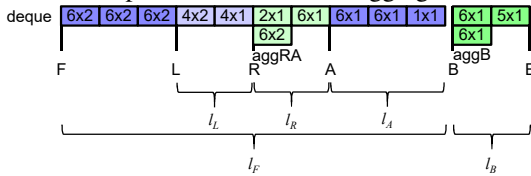
6 DABA Lite

DABA Lite improves upon the space complexity of DABA without increasing its running time, storing only $n + 2$ partial aggregates, compared to $2n$ in DABA. It saves space by exploiting the insights that the DABA algorithm reads none of the *val* fields of the sublists that are aggregated to the left and only the last *agg* fields of sublists that are aggregated to the right. The time complexity is still worst-case $O(1)$ invocations of \otimes per SWAG operation. The data-structure visualizations in this section are all taken from concrete example traces shown in Figs. 7 and 8.

DABA Lite Data Structure. The DABA Lite data structure comprises a double-ended queue *deque* of partial aggregates, two additional partial aggregates *aggRA* and *aggB*, and six pointers F, L, R, A, B , and E into the queue, see Definition 4. The pointers are always ordered as follows:

$$F \leq L \leq R \leq A \leq B \leq E$$

Here is an example with a max-count aggregation:



Conceptually, each pointer p corresponds to a sublist l_p . Blue sublists are aggregated to the left to facilitate eviction, with each element containing the partial aggregate starting from that element to the right end of its sublist. The elements of green sublists simply contain the corresponding window elements. The aggregates for the green sublists are included in *aggRA* and *aggB*.

DABA Lite Invariants. The *contents invariants* specify the contents of the *deque* and of *aggRA* and *aggB*. Let v_0, \dots, v_{n-1} be the current window contents. In the leftmost portion of

sublist l_F (the front sublist, in dark blue), each element holds an aggregate starting from that element to the right end of l_F . In sublist l_L (the left sublist, in light blue), each element holds an aggregate starting from that element to the right end of l_L . In sublist l_R (the right sublist, in light green), each element holds the corresponding window element, and if $L \neq R$ then *aggRA* holds the combined partial aggregate of l_R and l_A . In sublist l_A (the accumulator sublist, in dark blue), each element holds an aggregate starting from that element to the right end of l_A . In sublist l_B (the back sublist, in dark green), each element holds the corresponding window element, and *aggB* holds the aggregate of l_B . Formally:

$$\forall i \in 0 \dots L - F - 1 : *(F + i) = v_i \otimes \dots \otimes v_{B-F-1}$$

$$\text{and } \forall i \in L - F \dots R - F - 1 : *(F + i) = v_i \otimes \dots \otimes v_{R-F-1}$$

$$\text{and } \forall i \in R - F \dots A - F - 1 : *(F + i) = v_i$$

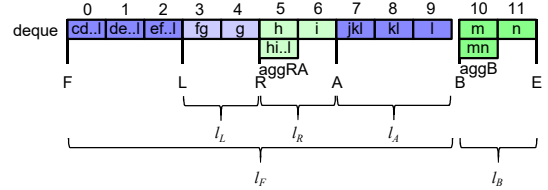
$$\text{and } (L = R) \vee (\text{aggRA} = v_{R-F} \otimes \dots \otimes v_{B-F-1})$$

$$\text{and } \forall i \in A - F \dots B - F - 1 : *(F + i) = v_i \otimes \dots \otimes v_{B-F-1}$$

$$\text{and } \forall i \in B - F \dots E - F - 1 : *(F + i) = v_i$$

$$\text{and } \text{aggB} = v_{B-F} \otimes \dots \otimes v_{E-F-1}$$

Here is a visual example of the invariants for a window with contents $v_0 = c, v_1 = d, \dots, v_{10} = m, v_{11} = n$:



The notation *cd..l* is shorthand for $c \otimes d \otimes \dots \otimes l$.

The *size invariants* specify constraints on the sizes of sublists. The size invariants of DABA Lite are the same as those of DABA:

$$(|l_F| = 0 \wedge |l_B| = 0) \vee (|l_L| + |l_R| + |l_A| + 1 = |l_F| - |l_B| \wedge |l_L| = |l_R|)$$

DABA Lite Algorithm. For each sublist l_p that is aggregated to the left, a private helper function Π_p^\otimes retrieves the corresponding partial aggregate or returns the monoid's identity element **1** if the sublist is empty.

```

1 fun  $\Pi_F^\otimes$ 
2   if ( $F = B$ ) return 1 else return  $*F$ 
3 fun  $\Pi_L^\otimes$ 
4   if ( $L = R$ ) return 1 else return  $*L$ 
5 fun  $\Pi_A^\otimes$ 
6   if ( $A = B$ ) return 1 else return  $*A$ 
```

These helpers return the correct values in constant time thanks to the invariants defined previously. Function *query* combines the aggregate of l_F and l_B , taking only a single invocation of \otimes .

```

7 fun query()
8   return  $\Pi_F^\otimes \otimes \text{aggB}$ 
```

Function *insert* pushes a value v onto l_B and updates *aggB* accordingly, then calls a function *fixup*, defined below, for doing one step of incremental reversal.

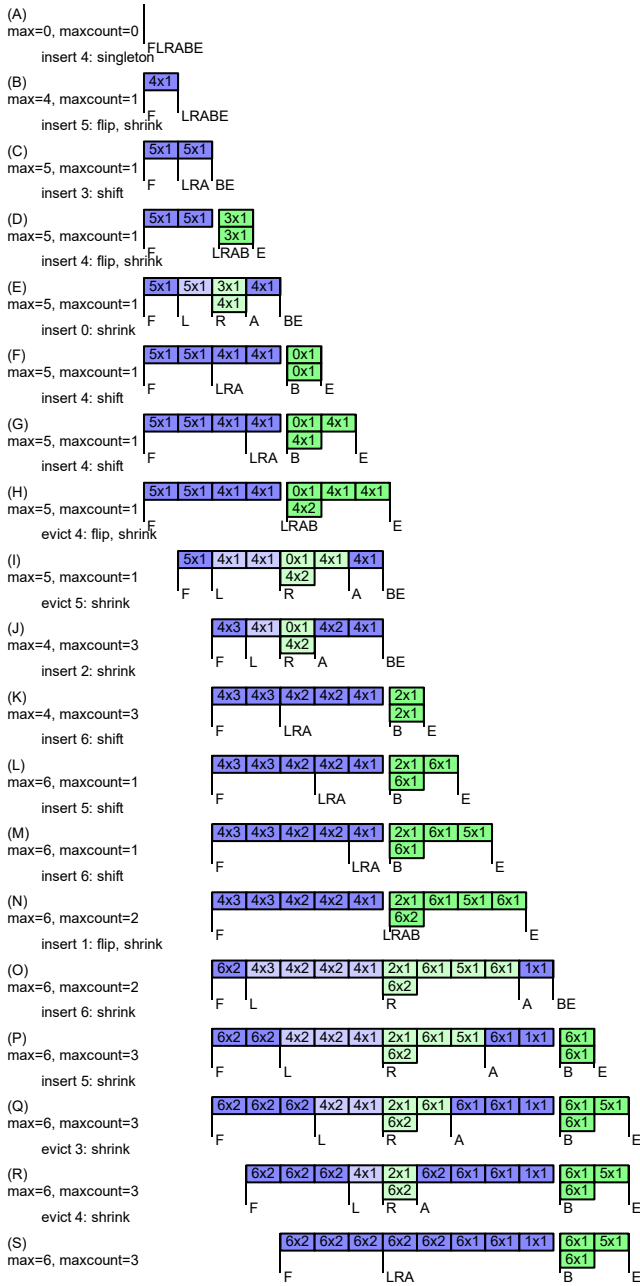


Fig. 7 DABA Lite example trace for maxcount aggregation. The notation $m \times c$ is shorthand for $\text{max}=m$, $\text{maxcount}=c$.

```

9 fun insert(v)
10   deque.pushBack(v)
11   E ← E + 1
12   aggB ← aggB ⊗ v
13   fixup()

```

In our running example, if $\text{procNameaggB} = 0 \otimes 1$ and $v = 4$, the newly pushed deque element is $4 \otimes 1$ and the updated aggB is also $4 \otimes 1$.

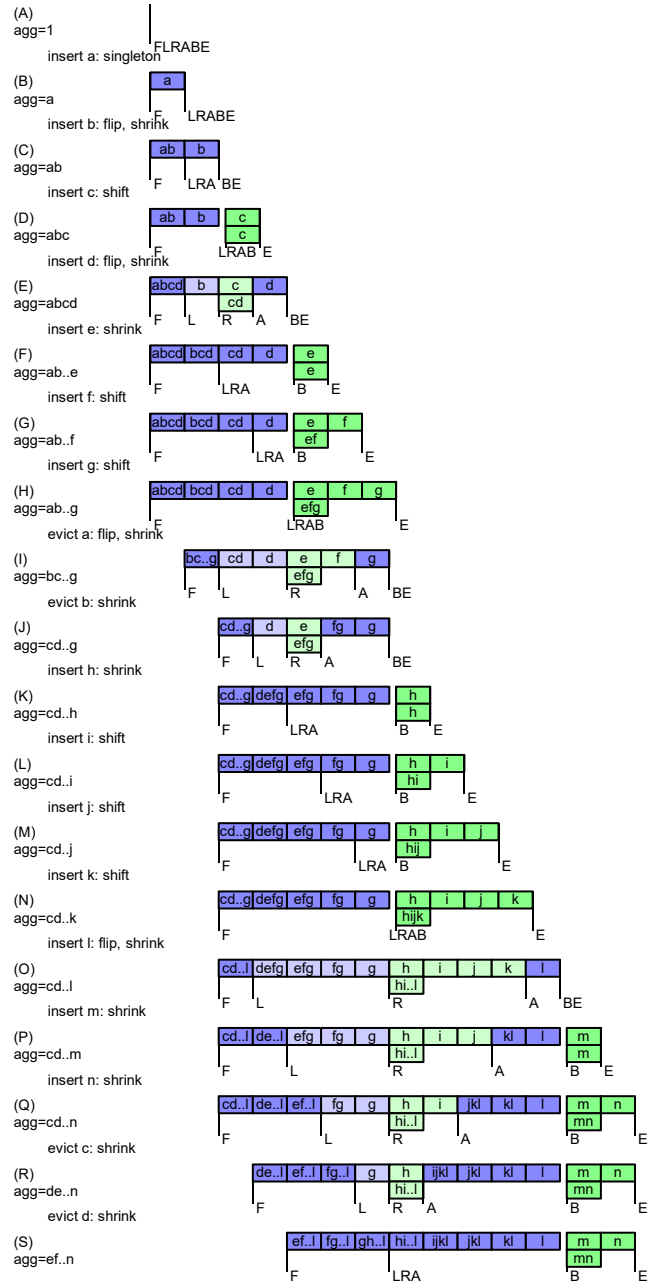
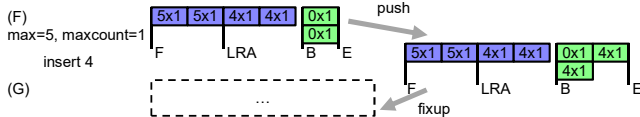


Fig. 8 DABA Lite example trace for any aggregation. The notation for aggregates omits \otimes , e.g., $bc..f$ is shorthand for $b \otimes c \dots \otimes f$.

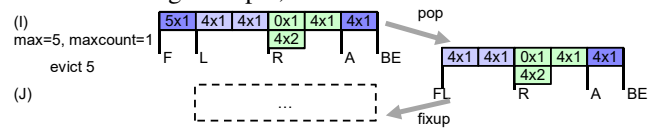
Similarly, *evict* pops an element from the front of the deque, then calls *fixup* for one step of incremental reversal.

```

14 fun evict()
15   F ← F + 1
16   deque.popFront()
17   fixup()

```

For our running example, eviction is illustrated below:



The *fixup* function repairs the invariants. The effect of *fixup* on the size invariants is the same for DABA and for DABA Lite. Since Section 5 has a formal analysis, here we only have an informal discussion. As before, the *fixup* function has four cases *singleton*, *flip*, *shift*, and *shrink*.

```

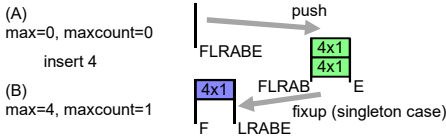
18 fun fixup()
19   if F = B # Singleton case
20     B ← E, A ← E, R ← E, L ← E
21     aggRA ← 1
22     aggB ← 1
23   else
24     if L = B # Flip
25       L ← F, A ← E, B ← E
26       aggRA ← aggB
27       aggB ← 1
28     if L = R # Shift
29       A ← A + 1, R ← R + 1, L ← L + 1
30   else # Shrink
31     *L ← ΠL⊗ ⊗ aggRA
32     L ← L + 1
33     *(A-1) ← *(A-1) ⊗ ΠA⊗
34     A ← A - 1

```

The *singleton* case happens when $|l_F| = 0$ and $|l_B| = 1$.

$B \leftarrow E, A \leftarrow E, R \leftarrow E, L \leftarrow E$

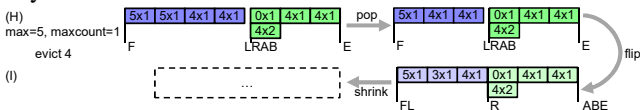
The pointer assignments change this to $|l_F| = 1$ and $|l_B| = 0$.



The *flip* case happens when $|l_F| > 0$ and $|l_L| = 0$ and $|l_R| = 0$ and $|l_A| = 0$. That implies that $|l_F| = |l_B|$, which means we can simply turn the old outer sublists l_F and l_B into the new inner sublists l_L and l_R .

$L \leftarrow F, A \leftarrow E, B \leftarrow E$
 $aggRA \leftarrow aggB$
 $aggB \leftarrow 1$

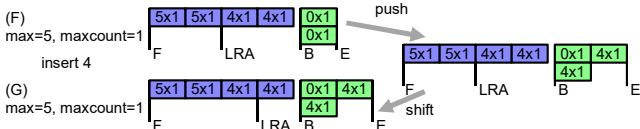
The corresponding updates to $aggRA$ and $aggB$ do not require any invocations of \otimes .



The *shift* case happens when $|l_F| > 0$ and $|l_L| = 0$. That means the pointers $L = R = A$ are equal, and can simply be moved one element to the right.

$A \leftarrow A + 1, R \leftarrow R + 1, L \leftarrow L + 1$

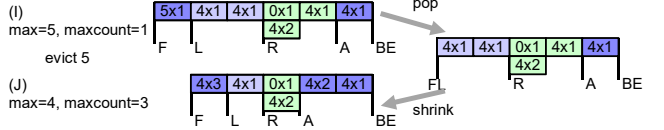
There is no need to update $aggRA$, since it will not be read anymore until after the next flip. The invariant for $aggRA$ remains satisfied, thanks to $L = R$.



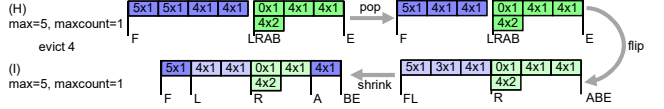
The *shrink* case happens when $|l_F| > 0$ and $|l_L| > 0$. The sizes of $|l_L| = |l_R|$ are the same, and shrink reduces them by one each. This requires setting agg fields of blue sublists appropriately for their contents invariants.

$*L \leftarrow \Pi_L^{\otimes} \otimes aggRA$
 $L \leftarrow L + 1$
 $*(A-1) \leftarrow *(A-1) \otimes \Pi_A^{\otimes}$
 $A \leftarrow A - 1$

Even though the internal boundary A between l_R and l_A moves, taken together, these two sublists still occupy the same elements, and thus, $aggRA$ does not change. Consequently, the shrink case of DABA Lite requires one less \otimes -invocation than the shrink case of DABA.



The shrink case also gets triggered right after a flip.



DABA Lite Theorems.

Lemma 11 *DABA Lite maintains both the contents invariants and the size invariants defined above.*

Proof The query function does not modify the data structure and thus does not change the invariants. The effect of *insert*, *evict*, and *fixup* on the size invariants is the same as for DABA. Whenever sublist boundaries change, the code updates the contents of *deque*, *aggRA*, and *aggB*, if necessary, to uphold the contents invariants. \square

Theorem 12 *If the window currently contains v_0, \dots, v_{n-1} , then query returns $v_0 \otimes \dots \otimes v_{n-1}$.*

Proof Using Lemma 11,

$query()$
 $= \Pi_F^{\otimes} \otimes aggB$
 $= v_0 \otimes \dots \otimes v_{B-F-1} \otimes v_{B-F} \otimes \dots \otimes v_{E-F-1}$
 $= v_0 \otimes \dots \otimes v_{n-1}$

\square

Theorem 13 *DABA Lite requires space to store $n + 2$ partial aggregates. DABA Lite invokes \otimes at most one time per query, three times per insert, and two times per evict. Furthermore, for nonempty windows, DABA Lite invokes \otimes on average two times per insert and one time per evict.*

Proof The algorithm contains no loops or recursion, so we can directly see the worst-case numbers from the code. The average-case numbers are based on the observation that every sequence of shrink steps is followed by an equal number of shift steps. Shrink requires two \otimes -invocations and shift requires zero \otimes -invocations, so the average *fixup* call has one \otimes -invocation. \square

7 Experimental Evaluation

The purpose of our experimental evaluation is to test whether DABA’s worst-case constant algorithmic complexity yields low latency and high throughput in practice, and to test whether DABA Lite is always more efficient than DABA.

Our experiments use six different SWAGs: Two-Stacks, Two-Stacks Lite, and FlatFIT [28] are all amortized $O(1)$, worst-case $O(n)$ algorithms designed for FIFO data. As originally published, FlatFIT does not support dynamic windows. We use a modified version that resizes FlatFIT’s circular buffer using the standard array doubling/shrinking technique without disturbing FlatFIT’s internal pointer structure. This results in an additional amortized $O(1)$ time per operation but supports dynamic windows and guarantees that the memory footprint is within a constant factor of the window size. We also adapted the published FlatFIT algorithm to our SWAG framework. DABA and DABA Lite are both worst-case $O(1)$ algorithms designed for FIFO data. FiBA [33] is designed for out-of-order data and reduces to amortized $O(1)$, worst-case $O(\log n)$ in the FIFO case. All of our experiments with FiBA use a min-arity of 4.

We chose three representative aggregation operators to span the execution cost spectrum. The operator `sum` is the sum of all items in the window, and it represents aggregation operations so cheap that the traversal and changes to the underlying data structure should dominate performance. The operator `bloom` applies a Bloom filter to all items in the window, and it represents aggregation operations where the operator itself dominates performance. The operator is so expensive that minimizing calls to it matters more than changes to the underlying data structure. Finally, `geomean` computes the geometric mean of all items in the window. It represents a middle ground of operator cost.

We implemented all algorithms in C++11, using the g++ compiler version 7.5.0 with optimization level `-O3`. Our system runs Red Hat 7.3, with Linux kernel version 3.10.0. The processor is an Intel Platinum 8168 at 2.7 GHz. All implementations, experiments, and post-processing scripts used in this section are available from the open-source project Sliding Window Aggregators².

7.1 Static Windows

The experiments in Figs 9 and 10 use a static count-based window with synthetic data. In each experiment, we first *insert* n data items, where n is the size of the window. The timed part of the experiment consists of rounds of *evict*, *insert*, and *query*. For the latency experiments, we record

² Available at <https://github.com/IBM/sliding-window-aggregators>. Our experiments use the C++ implementations and benchmarks, as well as the Python scripts from commit 41ee775.

all times for 10 million rounds with a fixed window size of 2^{14} data items. For the throughput experiments, we time how long it takes to complete 200 million rounds, and we vary n from 1 to 2^{22} .

The practical reason to choose a worst-case $O(1)$ aggregator is to minimize latency. Both Two-Stacks and Two-Stacks Lite in Fig 9 tend to have lower minimum latency than both DABA and DABA Lite. But, true to their linear worst-case, Two-Stacks and Two-Stacks Lite regularly suffer from an order-of-magnitude higher latency. This trend becomes more pronounced as the cost of the aggregation function increases. Unlike the other aggregators, FiBA is tree based. As maintaining the tree is more up-front work, it tends to have high minimum and median latency. But, also being tree based, its worst-case behavior is bounded by $O(\log n)$; it has lower worst-case latency than the worst-case $O(n)$ aggregators. FlatFIT is not a tree-based structure, but the access pattern during queries ends up having similar properties: successive indirect accesses to different parts of the window. Each query pushes indices onto a stack, and then pops indices from the stack to indirectly access the window.³ This is a large amount of work, and when the aggregation operation is cheap, this work dominates performance and yields a high latency floor.

All of the aggregators are able to maintain close to constant behavior in Fig 10, although there are large differences between them. Surprisingly, DABA’s throughput is more competitive with Two-Stacks than in prior work [31,32]. We attribute this difference to a more modern compiler with more aggressive inlining and dead-code elimination. Both DABA Lite and Two-Stacks Lite always outperform the corresponding non-Lite versions. FlatFIT becomes more competitive with expensive operations as its indirect accesses become less important compared to the total number of calls to the aggregation operation.

7.2 Dynamic Windows

The experiments in Fig. 11 use a dynamic count-based window with synthetic data. The experiments time a fill-and-drain pattern for a total of 200 million data items. It performs *insert* and *query* until reaching the window size n , and then calls *evict* until the window is down to 0, then repeats. We vary n from 1 to 2^{22} .

The throughput trends are largely the same as with static windows, which is the point of these experiments: even with dynamically changing window sizes, the fundamental properties of these streaming aggregation algorithms remain mostly

³ Our implementation performs an optimization where the same stack is reused across queries. This is safe because the stack is always empty at the end of a query. For dynamic windows, the number of indices involved can be non-constant. Avoiding the recreation of the stack and reusing the same memory makes about a 20% difference in throughput, but does not change FlatFIT’s overall comparative performance.

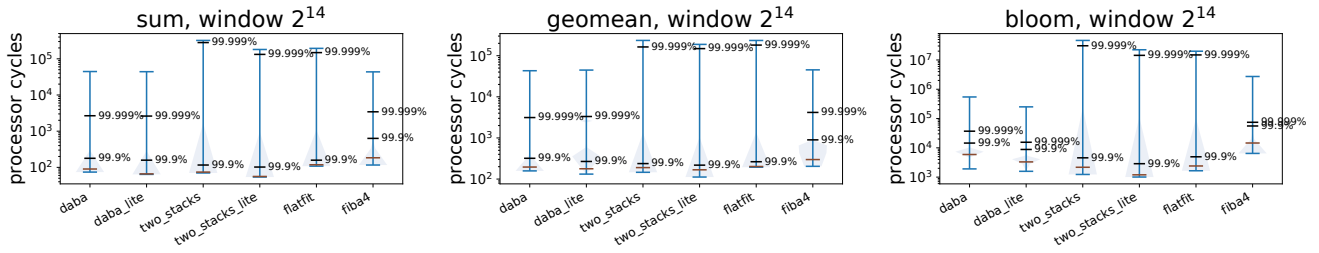


Fig. 9 Latency, shown as violin plots, for static count-based windows with synthetic data.

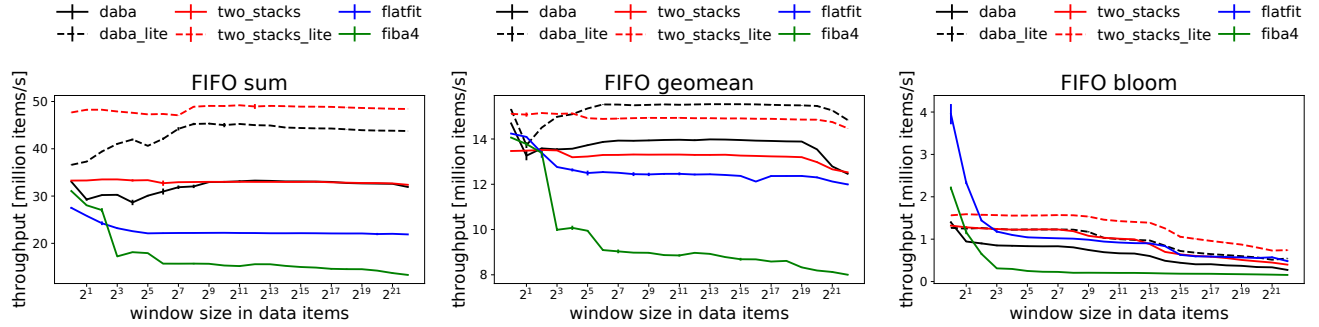


Fig. 10 Throughput for static count-based windows with synthetic data.

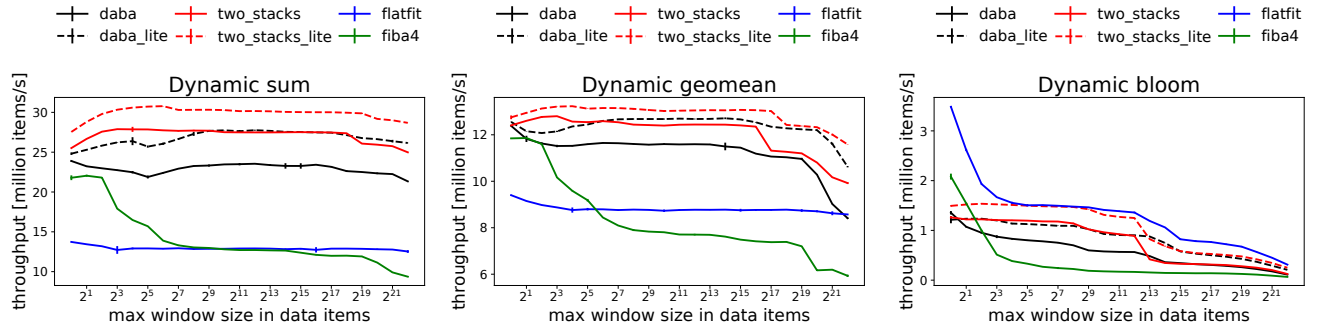


Fig. 11 Throughput for dynamic count-based windows with synthetic data.

the same. The one major difference is FlatFIT, whose throughput is consistently the best for bloom, which is the most expensive aggregation operation. This experimental design happens to be close to a best-case for FlatFIT, as it does not call the aggregation operation on evictions. FlatFIT only calls the aggregation operation on queries. The other algorithms require calling the aggregation operation on evictions in order to maintain their various properties of their partial aggregates. But, since the experiment performs no queries when it drains the window, such work is “wasted” in this case.

7.3 Real Data

The experiments in Fig. 12 use dynamic event-based windows based on real data. We use the dataset from the DEBS 2012 Grand Challenge [1], which recorded data from manufacturing equipment at approximately 100 Hz. We removed about 1.5% of the 32.3 million events to enforce in-order

data to make it suitable for FIFO aggregation algorithms. Our experiments maintain an event-time-based window of τ seconds, which means that the actual number of data items in that window will fluctuate over time. We do not start measuring until the window has evicted its first data item. In the throughput experiments, we vary τ from 10 milliseconds to 6 hours. In the latency experiments, we choose a window of 10 minutes. For both experiments, we use an aggregation operation inspired by Query 2 of the DEBS 2012 Grand Challenge: relative variation.

Because FiBA was designed for out-of-order data, it natively has a concept of timestamps. But the other algorithms are FIFO aggregators and do not natively support timestamps. These sets of experiments use modified versions of all of the other aggregators that add support to query the youngest and oldest timestamps in the window. We use those queries to maintain the event-time window of τ seconds.

The results are consistent with the previous experiments, with two exceptions. First, DABA, DABA Lite, Two-Stacks,

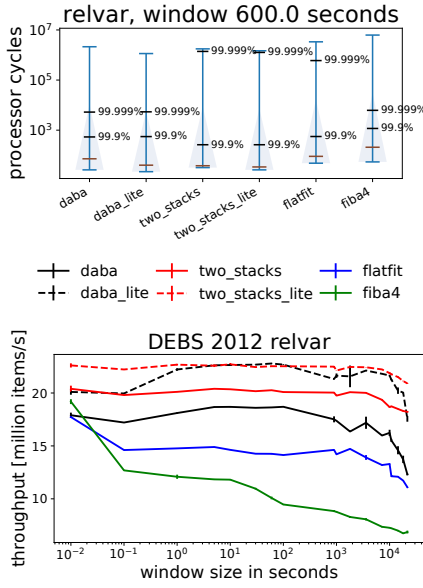


Fig. 12 Throughput and latency for dynamic event-based windows with the manufacturing equipment data set.

Two-Stacks Lite, and FlatFIT experience similar maximum latency, unlike with the static experiments. This latency similarity is caused by rare bulk evictions: rounds where more than 100 items are evicted experienced latencies more than $100\times$ greater than the previous round. Since maintaining the same window is a shared property across all experiments, all algorithms have similar maximum latency. Second, the throughput of all algorithms experiences some degradation after 10^3 seconds. This is because the actual window size is becoming a large enough fraction of the total data that rare events are not amortized.

7.4 Discussion

Our experiments demonstrate several consistent behaviors. The throughput of the Lite variants of both DABA and Two-Stacks are always significantly better than the original version, but the overall difference in latency is less dramatic. Two-Stacks Lite tends to have the best throughput. Since we lacked access to an implementation of FlatFIT by its authors, we re-implemented it based on their paper and extended it to support variable-sized windows. Our implementation of FlatFIT tends to have a high latency floor, but a median that is close to that floor. However, its maximum latency is consistent with being worst-case $O(n)$, and its throughput is only competitive with expensive operators or when the experiment happens to align with its design.

The theoretical analysis established that both DABA and DABA Lite are worst-case constant time algorithms. But low latency and competitive throughputs are sensitive to what the actual constant is. Our experiments demonstrate that

DABA and DABA Lite are able to realize low latency and competitive throughput across a large range of n with both static and dynamic windows and with real data.

8 Related Work

This section discusses the literature on sliding window aggregation algorithms with an emphasis on in-order streams and associative aggregation operators \otimes . As before, let n be the window size.

8.1 Solutions to the Same Problem

Section 2 formalized the problem statement as an abstract data type called SWAG for first-in-first-out sliding window aggregation. This section discusses concrete algorithms that implement the abstract data type. Section 2 phrased associative aggregation operators as monoids. While some monoids are invertible or commutative, that is not true for all monoids, and thus, this section only lists algorithms that work without such additional algebraic properties. This section presents algorithms in chronological order by publication date.

Recalculate-from-scratch implements SWAG by maintaining a FIFO queue of all data items and calculating the aggregation of the entire queue for each query. Each query requires $O(n)$ invocations of \otimes . The queue takes up space for n data stream values.

The *B-Int* algorithm from 2004 [7] implements SWAG using base intervals, which are similar to a balanced binary tree. The time complexity is $O(\log n)$ invocations of \otimes and the space complexity is around $2n$ partial aggregates: n for leaves plus $\sim n$ for internal nodes.

The *Two-Stacks* idea was mentioned in a Stack Overflow post from 2011 [3], which described the idea for one aggregation operator, minimum. Even though Two-Stacks is a SWAG algorithm, it was not immediately noticed as such by the academic community. As discussed in Section 3, the time complexity is amortized $O(1)$ invocations of \otimes with a worst-case of $O(n)$, and the space is $2n$ partial aggregates.

The *Reactive Aggregator* from 2015 implements SWAG using a perfect binary tree. This algorithm uses a data structure called FlatFAT, which stands for flat fixed-sized aggregator and represents a perfect binary tree without storing explicit pointers and without needing any rebalancing. The time complexity is amortized $O(\log n)$ with a worst-case of $O(n)$ invocations of \otimes . If n is a power of two, FlatFAT requires space for $\sim 2n$ partial aggregates: n leaves plus n interior nodes. If n is slightly above a power of two, the space can be up to $\sim 4n$.

The *DABA* algorithm was first published in 2015 [31]. DABA was inspired by Okasaki's purely functional queues and dequeues [26]. However, the two differ substantially: DABA

is not a purely functional data structure, and Okasaki’s data structures do not implement sliding window aggregation. As discussed in Section 5, DABA requires worst-case $O(1)$ invocations of \otimes and the space is $2n$ partial aggregates.

The *FlatFIT* algorithm from 2017 implements SWAG via a flat and fast index traverser [28]. The time complexity is amortized $O(1)$ invocations of \otimes with a worst-case of $O(n)$. The algorithm stores n partial aggregates as well as n pointers, which are indices into the window for stitching together the partial aggregates of subranges. The algorithm requires an additional stack of indices for pointer updates, and the authors report the total space requirements as up to $2.5n$.

The *Hammer Slide* paper from 2018 [35] starts from Two-Stacks and optimizes it further. The time complexity remains amortized $O(1)$ invocations of \otimes with a worst-case of $O(n)$. One of the optimizations from Hammer Slide is to only store $n + 1$ partial aggregates by observing what is needed for the front stack and the back stack. The Two-Stacks Lite algorithm in Section 4 takes inspiration from Hammer Slide.

The *AMTA* algorithm from 2019 [40] implements SWAG via an amortized monoid tree aggregator. AMTA adds sophisticated tree representations that optimize FIFO insert and evict. Its amortized algorithmic time complexity is $O(1)$ invocations of \otimes , with a worst-case of $O(\log n)$. Like other tree-based SWAGs, AMTA requires $\sim 2n$ space for n leaves and $\sim n$ inner nodes. KVS-AMTA is an out-of-memory variant that externalizes most of this space into a key-value store.

The *FiBA* algorithm from 2019 [33] implements SWAG via a finger B-tree aggregator. FiBA uses finger pointers, position-aware partial aggregates, and a suitable rebalancing strategy to optimize insert and evict near the start and end of the window. For the FIFO case, its amortized algorithmic time complexity is $O(1)$ invocations of \otimes , with a worst-case of $O(\log n)$. Its space complexity depends on the arity of the B-tree. Since the minimum arity of B-trees is more than binary, B-trees store fewer than $\sim 2n$ partial aggregates.

The *DABA Lite* algorithm has not been published before, making it an original contribution of this paper. As discussed in Section 6, the time complexity is worst-case $O(1)$ invocations of \otimes and the space is $n + 2$ partial aggregates.

8.2 Complementary Techniques

While Section 2 states the core problem for sliding-window aggregation, there are often additional requirements. This section discusses techniques for augmenting algorithms that implement the SWAG abstract data type (including DABA and DABA Lite) to solve a broader set of problems.

Coarse-grained sliding reduces the effective window size n by storing only a single partial aggregate for values that will be evicted together. A state-of-the-art algorithm for coarse-grained sliding is Scotty [38]. Reducing n reduces the time

complexity of any algorithms whose time complexity depends upon n . Being worst-case $O(1)$, the time complexity of DABA and DABA Lite does not depend on n . Reducing n also reduces the space complexity, which is somewhere between n and $\sim 4n$ for all SWAG algorithms from Section 8.1.

Bounded disorder handling tolerates out-of-order arrivals of data stream items as long as the disorder is not too large. Srivastava and Widom described how to handle bounded disorder by buffering incoming data items [30]. Later, when data items are released from the buffer, they are ordered by their nominal timestamps. That makes it possible to use in-order SWAG algorithms from Section 8.1.

Partition parallelism is a way to parallelize stateful streaming applications as long as the computation for each partition key is independent from the computation for the other keys [27]. Sliding-window aggregation is often used in a way that satisfies this requirement, by aggregating separately within each key. In that case, parallelization can just maintain separate instances of a given SWAG. For this to work well, it is best not to conservatively preallocate too much memory, lest the data structures for rare keys take up too much space.

Given various algorithms and techniques, how can we pick and combine the right ones for a given problem? A recent paper by Traub et al. [39] presents decision trees for dispatching to the right combination given the stream order, window kinds, aggregation operators, window sharing, etc. We argue that DABA Lite should be used for the in-order case with associative aggregation operators.

8.3 Solutions to Other Problems

Of course there are also problems around sliding-window aggregations where it does not suffice to just combine a SWAG algorithm from Section 8.1 with a complementary technique from Section 8.2. This section highlights a few such problems with solutions; for more details see [18].

Window sharing serves sliding window aggregation queries for multiple window sizes from a single data structure. Not all data structures are suitable for this. SWAG algorithms from Section 8.1 that support window sharing include B-Int [7], FlatFIT [28], and FiBA [33]. The SlideSlide algorithm implements SWAG for fixed-sized windows [36].

Unbounded disorder handling tolerates out-of-order arrivals that are arbitrarily late, incorporating them into the data structure whenever they arrive. Truviso accomplishes this for the case where multiple input streams have drifted arbitrarily far from each other, as long as each of the input streams is internally in-order [21]. FiBA supports general out-of-order sliding window aggregation without restrictions on the degree of disorder [33]. The algorithmic complexity of FiBA matches the theoretical lower bound for this problem.

When it comes to *aggregation operators*, some algorithms are more restrictive than our problem statement from

Section 2. For instance, subtract-on-evict is a simple algorithm that only works when subtraction is well-defined, in other words, when the \otimes operator is invertible. Similarly, SlickDeque [29] only works for aggregation operators that are either invertible or that satisfy the property that $x \otimes y \in \{x, y\}$. On the other hand, there are also some aggregation operators for which our problem statement from Section 2 is a poor fit. One of the most prominent ones is median, or more generally, percentile aggregation. An efficient solution for sliding-window median and percentiles is an order statistics tree [17].

9 Conclusion

This paper is a journal version of our earlier conference paper [32] about DABA, the first algorithm for in-order sliding window aggregation in worst-case constant time. Besides providing a more comprehensive description of DABA, this paper also introduces a new algorithm called DABA Lite that improves over DABA. Where DABA requires space to store $2n$ partial aggregates, DABA Lite only stores $n + 2$ partial aggregates. Whereas DABA requires on average 2.5 invocations of the underlying monoid per insert and 1.5 per evict, DABA Lite requires on average only 2 invocations per insert and 1 per evict. The worst-case time complexity is constant just like for DABA.

DABA and DABA Lite have several desirable properties. They only require an associative monoid (no need for commutativity nor invertibility). They support dynamically-sized windows, where the window size can fluctuate throughout the execution, for instance, due to a variable interarrival rate of stream data items. They are built on a simple flat data structure, thus avoiding memory-copy or allocation churn, as well as avoiding excessive pointer chasing. Our experiments demonstrate that DABA Lite performs well compared to other sliding-window aggregation algorithms.

References

1. DEBS 2012 Grand Challenge: Manufacturing equipment. <https://debs.org/grand-challenges/2012>. Retrieved June 2020
2. Apache Flink: Scalable batch and stream data processing. <https://flink.apache.org> (2016). Retrieved Aug. 2016
3. adamax: Re: Implement a queue in which `push_rear()`, `pop_front()` and `get_min()` are all constant time operations. <http://stackoverflow.com/questions/4802038/> (2011). Retrieved Aug., 2016
4. Agarwal, P.K., Cormode, G., Huang, Z., Phillips, J., Wei, Z., Yi, K.: Mergeable summaries. In: Symposium on Principles of Database Systems (PODS), pp. 23–34 (2012)
5. Akidau, T., Balikov, A., Bekiroglu, K., Chernyak, S., Haberman, J., Lax, R., McVeety, S., Mills, D., Nordstrom, P., Whittle, S.: MillWheel: Fault-tolerant stream processing at internet scale. In: Conference on Very Large Data Bases (VLDB) Industrial Track, pp. 734–746 (2013)
6. Ali, M., Chandramouli, B., Goldstein, J., Schindlauer, R.: The extensibility framework in Microsoft StreamInsight. In: International Conference on Data Engineering (ICDE), pp. 1242–1253 (2011)
7. Arasu, A., Widom, J.: Resource sharing in continuous sliding window aggregates. In: Conference on Very Large Data Bases (VLDB), pp. 336–347 (2004)
8. Bloom, B.H.: Space/time trade-offs in hash coding with allowable errors. *Communications of the ACM (CACM)* **13**(7), 422–426 (1970)
9. Boykin, O., Ritchie, S., O’Connell, I., Lin, J.: Summingbird: A framework for integrating batch and online MapReduce computations. In: Conference on Very Large Data Bases (VLDB), pp. 1441–1451 (2014)
10. Carbone, P., Traub, J., Katsifodimos, A., Haridi, S., Markl, V.: Cutty: Aggregate sharing for user-defined windows. In: Conference on Information and Knowledge Management (CIKM), pp. 1201–1210 (2016)
11. Cormen, T.H., Leiserson, C.E., Rivest, R.L., Stein, C.: Introduction to Algorithms, 3rd Edition. MIT Press (2009). URL <http://mitpress.mit.edu/books/introduction-to-algorithms>
12. Cormode, G., Muthukrishnan, S.: An improved data stream summary: The count-min sketch and its applications. *Journal of Algorithms* **55**(1), 58–75 (2005)
13. Cranor, C., Johnson, T., Spataschek, O., Shkapenyuk, V.: Gigascope: A stream database for network applications. In: International Conference on Management of Data (SIGMOD) Industrial Track, pp. 647–651 (2003)
14. Flajolet, P., Fusy, E., Gandouet, O., Meunier, F.: HyperLogLog: The analysis of a near-optimal cardinality estimation algorithm. In: Conference on Analysis of Algorithms (AofA), pp. 127–146 (2007)
15. Gedik, B.: Generic windowing support for extensible stream processing systems. *Software Practice and Experience (SP&E)* pp. 1105–1128 (2013)
16. Hirzel, M., Andrade, H., Gedik, B., Jacques-Silva, G., Khandekar, R., Kumar, V., Mendell, M., Nasaard, H., Schneider, S., Soulé, R., Wu, K.L.: IBM Streams Processing Language: Analyzing big data in motion. *IBM Journal of Research and Development* **57**(3/4) (2013)
17. Hirzel, M., Rabbah, R., Suter, P., Tardieu, O., Vaziri, M.: Spreadsheets for stream processing with unbounded windows and partitions. In: Conference on Distributed Event-Based Systems (DEBS), pp. 49–60 (2016)
18. Hirzel, M., Schneider, S., Tangwongsan, K.: Tutorial: Sliding-window aggregation algorithms. In: Conference on Distributed Event-Based Systems (DEBS), pp. 11–14 (2017)
19. Izbicki, M.: Algebraic classifiers: A generic approach to fast cross-validation, online training, and parallel training. In: International Conference on Machine Learning (ICML), pp. 648–656 (2013)
20. Jügel, U., Jerzak, Z., Hackenbroich, G., Markl, V.: M4: A visualization-oriented time series data aggregation. In: Conference on Very Large Data Bases (VLDB), pp. 797–808 (2014)
21. Krishnamurthy, S., Franklin, M.J., Davis, J., Farina, D., Golovko, P., Li, A., Thombre, N.: Continuous analytics over discontinuous streams. In: International Conference on Management of Data (SIGMOD), pp. 1081–1092 (2010)
22. Krishnamurthy, S., Wu, C., Franklin, M.: On-the-fly sharing for streamed aggregation. In: International Conference on Management of Data (SIGMOD), pp. 623–634 (2006)
23. Kulkarni, S., Bhagat, N., Fu, M., Kedigehalli, V., Kellogg, C., Mittal, S., Patel, J.M., Ramasamy, K., Taneja, S.: Twitter Heron: Stream processing at scale. In: International Conference on Management of Data (SIGMOD), pp. 239–250 (2015)
24. Li, J., Maier, D., Tufte, K., Papadimos, V., Tucker, P.A.: No pane, no gain: efficient evaluation of sliding-window aggregates over data streams. *ACM SIGMOD Record* **34**(1), 39–44 (2005)

25. Murray, D.G., McSherry, F., Isaacs, R., Isard, M., Barham, P., Abadi, M.: Naiad: A timely dataflow system. In: Symposium on Operating Systems Principles (SOSP) (2013)
26. Okasaki, C.: Simple and efficient purely functional queues and dequeues. *Journal of Functional Programming (JFP)* **5**(4), 583–592 (1995)
27. Schneider, S., Hirzel, M., Gedik, B., Wu, K.L.: Safe data parallelism for general streaming. *IEEE Transactions on Computers (TC)* **64**(2), 504–517 (2015)
28. Shein, A.U., Chrysanthos, P.K., Labrinidis, A.: FlatFIT: Accelerated incremental sliding-window aggregation for real-time analytics. In: Conference on Scientific and Statistical Database Management (SSDBM), pp. 5.1–5.12 (2017)
29. Shein, A.U., Chrysanthos, P.K., Labrinidis, A.: SlickDeque: High throughput and low latency incremental sliding-window aggregation. In: Conference on Extending Database Technology (EDBT), pp. 397–408 (2018)
30. Srivastava, U., Widom, J.: Flexible time management in data stream systems. In: Principles of Database Systems (PODS), pp. 263–274 (2004)
31. Tangwongsan, K., Hirzel, M., Schneider, S.: Constant-time sliding window aggregation. Tech. Rep. RC25574, IBM Research (2015)
32. Tangwongsan, K., Hirzel, M., Schneider, S.: Low-latency sliding-window aggregation in worst-case constant time. In: Conference on Distributed Event-Based Systems (DEBS), pp. 66–77 (2017)
33. Tangwongsan, K., Hirzel, M., Schneider, S.: Optimal and general out-of-order sliding-window aggregation. In: Conference on Very Large Data Bases (VLDB), pp. 1167–1180 (2019)
34. Tangwongsan, K., Hirzel, M., Schneider, S., Wu, K.L.: General incremental sliding-window aggregation. In: Conference on Very Large Data Bases (VLDB), pp. 702–713 (2015)
35. Theodorakis, G., Kolios, A., Pietzuch, P.R., Pirk, H.: Hammer Slide: Work- and CPU-efficient streaming window aggregation. In: Workshop on Accelerating Analytics and Data Management Systems Using Modern Processor and Storage Architectures (ADMS), pp. 34–41 (2018)
36. Theodorakis, G., Pietzuch, P.R., Pirk, H.: SlideSlide: A fast incremental stream processing algorithm for multiple queries. In: Conference on Extending Database Technology (EDBT), pp. 435–438 (2020)
37. Toshniwal, A., Taneja, S., Shukla, A., Ramasamy, K., Patel, J.M., Kulkarni, S., Jackson, J., Gade, K., Fu, M., Donham, J., Bhagat, N., Mittal, S., Ryaboy, D.: Storm @Twitter. In: International Conference on Management of Data (SIGMOD), pp. 147–156 (2014)
38. Traub, J., Grulich, P., Cuellar, A.R., Breš, S., Katsifodimos, A., Rabl, T., Markl, V.: Scotty: Efficient window aggregation for out-of-order stream processing. In: Poster at the International Conference on Data Engineering (ICDE-Poster) (2018)
39. Traub, J., Grulich, P., Cuellar, A.R., Breš, S., Katsifodimos, A., Rabl, T., Markl, V.: Efficient window aggregation with general stream slicing. In: Conference on Extending Database Technology (EDBT) (2019)
40. Villalba, A., Berral, J.L., Carrera, D.: Constant-time sliding window framework with reduced memory footprint and efficient bulk evictions. *Transactions on Parallel and Distributed Systems (TPDS)* **30**(3), 486–500 (2019)
41. Yu, Y., Gunda, P.K., Isard, M.: Distributed aggregation for data-parallel computing: Interfaces and implementations. In: Symposium on Operating Systems Principles (SOSP), pp. 247–260 (2009)
42. Zaharia, M., Das, T., Li, H., Hunter, T., Shenker, S., Stoica, I.: Discretized streams: Fault-tolerant streaming computation at scale. In: Symposium on Operating Systems Principles (SOSP), pp. 423–438 (2013)